

Derivatives of products and quotients

OVERVIEW: In this section, we derive formulas for derivatives of functions that are constructed by taking products and quotients of other functions, and we use these formulas to study rates of change in a variety of applications.

Topics:

- **The Product Rule**
- **Related-rate problems**
- **The Quotient Rule**
- **The derivative of $y = x^n$ for integers $n \geq 2$ and Mathematical Induction**

The rate of change of the area of a rectangle

Imagine that the sides of the rectangle in Figure 1 are changing, so that the width $w = w(t)$, height $h = h(t)$, and area $A(t) = w(t)h(t)$ of the rectangle are functions of the time t . We want a formula for the rate of change of the area in terms of w , h , and their rates of change.

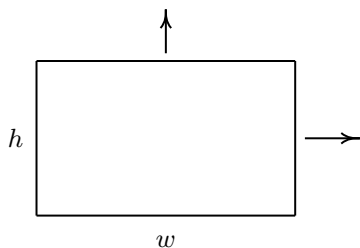


FIGURE 1

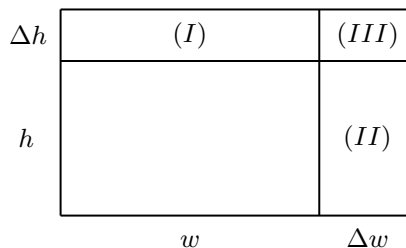


FIGURE 2

We consider a nonzero change Δt in the time from t to $t + \Delta t$ and let Δw and Δh be the corresponding changes in the width and height.

If Δw and Δh are positive, then at time $t + \Delta t$, the width is $w + \Delta w$ and the height is $h + \Delta h$, as in Figure 2, and the change ΔA in the area from t to $t + \Delta t$ is the area of the three rectangles labeled (I), (II), and (III) in Figure 2.

Example 1 Express the areas of rectangles (I), (II), and (III) in Figure 2 in terms of w , h , Δw , and Δh .

SOLUTION Rectangle (I) is w units wide and Δh units high and has area $w\Delta h$. Rectangle (II) is Δw units wide and h units high and has area $h\Delta w$. Rectangle (III) is Δw units wide and Δh units high and has area $\Delta w\Delta h$. \square

The results of Example 1 show that if Δw and Δh are positive, then

$$\Delta A = w\Delta h + h\Delta w + \Delta w\Delta h. \quad (1)$$

To verify (1) in general, we derive it algebraically. The area of a rectangle of width w and height h is wh , and the area of a rectangle of width $w + \Delta w$ and height $h + \Delta h$ is $(w + \Delta w)(h + \Delta h)$, so the change in the area is

$$\begin{aligned} \Delta A &= (w + \Delta w)(h + \Delta h) - wh \\ &= (wh + w\Delta h + h\Delta w + \Delta w\Delta h) - wh \\ &= w\Delta h + h\Delta w + \Delta w\Delta h. \end{aligned}$$

This establishes equation (1). We divide both sides of it by the change Δt in the time to obtain

$$\frac{\Delta A}{\Delta t} = w \frac{\Delta h}{\Delta t} + h \frac{\Delta w}{\Delta t} + \Delta w \left[\frac{\Delta h}{\Delta t} \right]. \quad (2)$$

We suppose that $w = w(t)$ and $h = h(t)$ have derivatives at t . Then $w'(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta w}{\Delta t}$ and $h'(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta h}{\Delta t}$. Also $\Delta w \rightarrow 0$ as $\Delta t \rightarrow 0$ since, by Theorem 1 of Section 2.5, $w = w(t)$ is continuous at t . Hence (2) gives

$$\begin{aligned} A'(t) &= \lim_{\Delta t \rightarrow 0} \frac{\Delta A}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \left(w \frac{\Delta h}{\Delta t} + h \frac{\Delta w}{\Delta t} + \Delta w \frac{\Delta h}{\Delta t} \right) \\ &= w(t)h'(t) + h(t)w'(t) + (0)h'(t) \\ &= w(t)h'(t) + h(t)w'(t). \end{aligned} \quad (3)$$

Formula (3) is the PRODUCT RULE. It is restated in the following theorem with t, w , and h replaced by x, f , and g , and A replaced by fg :

Theorem 1 (The Product Rule) If $y = f(x)$ and $y = g(x)$ have derivatives at x , then so does their product, $y = f(x)g(x)$, and

$$(fg)' = fg' + gf' \quad (4a)$$

or with Leibniz notation,

$$\frac{d}{dx}(fg) = f \frac{dg}{dx} + g \frac{df}{dx}. \quad (4b)$$

Remember the Product Rule as the following statement: the derivative of a product of two functions equals the first function multiplied by the derivative of the second, plus the second function multiplied by the derivative of the first.

Example 2 Find the derivative of $y = (x^5 + x^2)(x^{1/3} + 1)$ at $x = 1$.

SOLUTION By the Product Rule,

$$\begin{aligned} y'(x) &= \frac{d}{dx}[(x^5 + x^2)(x^{1/3} + 1)] \\ &= (x^5 + x^2) \frac{d}{dx}(x^{1/3} + 1) + (x^{1/3} + 1) \frac{d}{dx}(x^5 + x^2) \\ &= (x^5 + x^2) \left(\frac{1}{3} x^{-2/3} \right) + (x^{1/3} + 1)(5x^4 + 2x). \end{aligned}$$

Therefore, $y'(1) = (1 + 1)\left(\frac{1}{3}\right) + (1 + 1)(5 + 2) = 14\frac{2}{3}$. \square .

Example 3 Find the rate of change of the area of a rectangle at a moment when the width is 4 meters, the height is 2 meters, the width is increasing 3 meters per hour, and the height is increasing 5 meters per hour.

SOLUTION Equation (3) states that $A' = wh' + hw'$. We set $w = 4$, $h = 2$, $w' = 3$, and $h' = 5$ to obtain

$$\begin{aligned} A' &= [4 \text{ meters}] \left[5 \frac{\text{meters}}{\text{hour}} \right] + [2 \text{ meters}] \left[3 \frac{\text{meters}}{\text{hour}} \right] \\ &= 4(5) + 2(3) = 26 \frac{\text{square meters}}{\text{hour}}. \quad \square \end{aligned}$$

Related-rate problems

In solving Example 3 we started with an equation, $A = wh$, relating three functions of time, w , h , and A . We differentiated the equation with respect to t to obtain an equation, $A' = wh' + hw'$, which we used to find the rate of change of A from w , h , and their rates of change. This type of problem, which involves rates of change of related functions, is called a RELATED-RATE PROBLEM. Here is another example.

Example 4 At the beginning of 1990 the total population of the U.S was 248.7 million, of whom 51.3% were women, the total population was increasing at the rate of 3.5 million per year, and the percentage of women was decreasing 0.04% per year.⁽¹⁾ At what rate was the population of women increasing at the beginning of 1990?

SOLUTION We let $p = p(t)$ be the total U.S. population (measured in millions) in year t and let $F = F(t)$ be the fraction that were women (the percent divided by 100). Then the population of women at time t is $W(t) = F(t)p(t)$, so that by the Product Rule, the rate of change of the population of women is $W'(t) = F(t)p'(t) + F'(t)p(t)$.

Since $p(1990) = 248.7$, $F(1990) = 0.513$, $p'(1990) = 3.5$, and $F'(1990) = -0.0004$, we obtain

$$\begin{aligned} W'(1990) &= F(1990)p'(1990) + F'(1990)p(1990) \\ &= (0.513)(3.5) + (248.7)(-0.0004) \doteq 1.7. \end{aligned}$$

At the beginning of 1990 the population of women was increasing at the rate of approximately 1.7 million per year. \square

⁽¹⁾Data adapted from *1993 Statistical Abstract of the United States*, Washington, DC: U. S. Department of Commerce, 1993, p. 15.

The Quotient Rule

Derivatives of quotients of functions are found by using the following theorem.

Theorem 2 (The Quotient Rule) At any value of x where $y = f(x)$ and $y = g(x)$ have derivatives and $g(x)$ is not zero, $y = f(x)/g(x)$ also has a derivative and

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2} \quad (5a)$$

or with Leibniz notation for the derivatives

$$\frac{d}{dx} \left(\frac{f}{g}\right) = \frac{g \frac{df}{dx} - f \frac{dg}{dx}}{g^2}. \quad (5b)$$

Remember this result as the following statement: the derivative of a quotient equals the denominator multiplied by the derivative of the numerator, minus the numerator multiplied by the derivative of the denominator, all divided by the square of the denominator.

Proof of Theorem 2: We consider first the case where f is the constant function 1. We write g for $g(x)$ with a fixed x where g is not zero and has a derivative. We let Δx denote a small, nonzero change in the variable, and we let $\Delta g = g(x + \Delta x) - g(x)$ be the resulting change g . Since g is continuous at x , $g(x + \Delta x)$ is not zero for small nonzero Δx and tends to $g(x)$ as $\Delta x \rightarrow 0$. By the definition of the derivative,

$$\left. \frac{d}{dx} \left(\frac{1}{g}\right) \right|_x = \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{g(x + \Delta x)} - \frac{1}{g(x)}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{g + \Delta g} - \frac{1}{g}}{\Delta x}. \quad (6)$$

We replace division by Δx with multiplication by $1/(\Delta x)$ and take a common denominator:

$$\begin{aligned} \frac{\frac{1}{g + \Delta g} - \frac{1}{g}}{\Delta x} &= \frac{1}{\Delta x} \left(\frac{1}{g + \Delta g} - \frac{1}{g} \right) = \frac{1}{\Delta x} \left(\frac{g - (g + \Delta g)}{(g + \Delta g)g} \right) \\ &= \frac{1}{\Delta x} \left(\frac{-\Delta g}{(g + \Delta g)g} \right) = \left(\frac{-1}{(g + \Delta g)g} \right) \frac{\Delta g}{\Delta x}. \end{aligned}$$

Since $\Delta g = g(x + \Delta x) - g(x)$ tends to 0 and $\Delta g/\Delta x$ tends to the derivative $g' = g'(x)$ as $\Delta x \rightarrow 0$, the last equation with (6) gives

$$\left. \frac{d}{dx} \left(\frac{1}{g}\right) \right|_x = \lim_{\Delta x \rightarrow 0} \left(\frac{-1}{(g + \Delta g)g} \frac{\Delta g}{\Delta x} \right) = \frac{-g'}{g^2}. \quad (7)$$

This is (5a) with $f = 1$ since $f' = 0$.

We can now derive (5a) for a general f that has a derivative at x by using (7) and the Product Rule. We obtain

$$\begin{aligned} \frac{d}{dx} \left(\frac{f}{g}\right) &= \frac{d}{dx} \left[\left(\frac{1}{g}\right) f \right] = \left(\frac{1}{g}\right) f' + f \frac{d}{dx} \left(\frac{1}{g}\right) \\ &= \frac{f'}{g} + f \left(\frac{-g'}{g^2}\right) = \frac{gf' - fg'}{g^2}. \quad \text{QED} \end{aligned}$$

Example 5 What is $R'(0)$ if $R(x) = P(x)/Q(x)$, $P(0) = 3$, $P'(0) = 10$, $Q(0) = 5$, and $Q'(0) = 50$?

SOLUTION By (5a) with P and Q in place of f and g ,

$$R'(0) = \frac{Q(0)P'(0) - P(0)Q'(0)}{[Q(0)]^2} = \frac{5(10) - 3(50)}{5^2} = \frac{50 - 150}{25} = -4. \quad \square$$

Example 6 What is $\frac{dy}{dx}$ if $y = \frac{x^2}{x^4 + a}$ with constant a ?

SOLUTION For all x such that $x^4 + a \neq 0$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left[\frac{x^2}{x^4 + a} \right] \\ &= \frac{(x^4 + a) \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(x^4 + a)}{(x^4 + a)^2} \\ &= \frac{(x^4 + a)(2x) - x^2(4x^3)}{(x^4 + a)^2} = \frac{2x^5 + 2ax - 4x^5}{(x^4 + a)^2} = \frac{2ax - 2x^5}{(x^4 + a)^2}. \quad \square \end{aligned}$$

Example 7 Figures 3 and 4 show graphs of the U.S. national debt $D = D(t)$ and the U. S. population $P = P(t)$ as functions of time from 1950 through 1990.⁽²⁾ (a) Find approximate values of $D(1985)$ and $P(1985)$ from the graphs. Then draw approximate tangent lines and estimate their slopes to find approximate values of $D'(1985)$ and $P'(1985)$. (b) Use your estimates from part (a) to give the approximate debt per person and the approximate rate of increase with respect to time of the debt per person at the beginning of 1985.

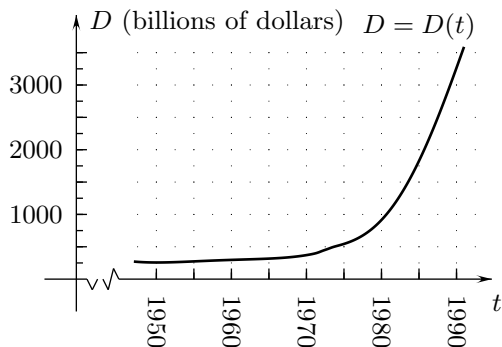


FIGURE 3

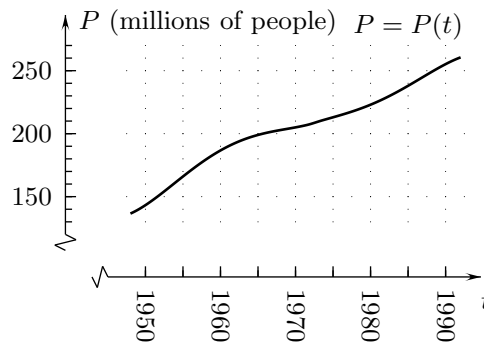


FIGURE 4

⁽²⁾Data adapted from *1993 Statistical Abstract of the United States*, Washington, DC: U. S. Department of Commerce, 1993, pp. 15 and 328.

SOLUTION

(a) The graphs show that $D(1985) \approx 1850$ billion dollars and $P(1985) \approx 240$ million people. The points (1980, 600) and (1985, 1850) are on the approximate tangent line in Figure 5. Therefore, $D'(1985) \approx \frac{1850 - 600}{1985 - 1980} \doteq 250$ billion dollars per year. Also the points (1970, 180) and (1985, 240) are on the approximate tangent line in Figure 6 so that $P'(1985) \approx \frac{240 - 180}{1985 - 1970} = 4$ million people per year.

(b) The debt per person at the beginning of 1985 was $\frac{D}{P} \approx \frac{1850 \text{ billion dollars}}{240 \text{ million people}} \doteq 7.7$ thousand dollars per person. The rate of change of the debt per person was

$$\begin{aligned} \frac{d}{dt} \left[\frac{D}{P} \right] &= \frac{PD' - DP'}{P^2} \approx \frac{(240)(250) - (1850)(4)}{240^2} \\ &\doteq 0.913 \text{ thousand dollars per person per year.} \end{aligned}$$

(Different estimates yield different answers.) \square

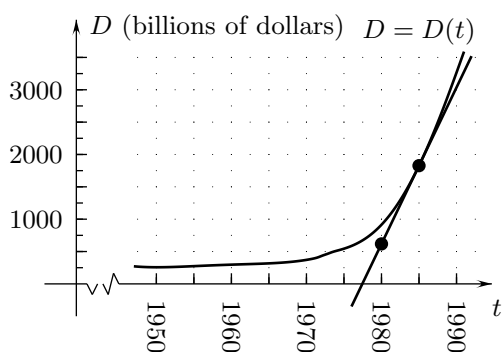


FIGURE 5

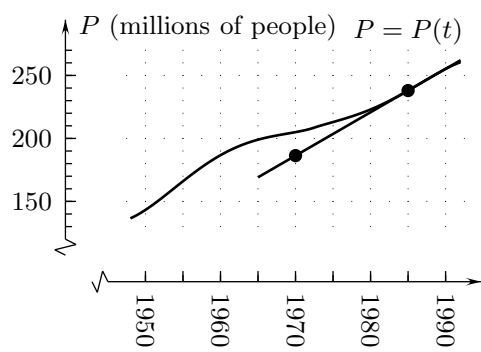


FIGURE 6

The derivative of $y = x^n$ for integers $n \geq 2$ and Mathematical Induction

In Section 2.4 we used the binomial theorem to derive the formula

$$\frac{d}{dx}(x^n) = nx^{n-1} \quad (8)$$

for positive integers n . Here we will prove that (8) holds for all such n by using its validity for $n = 1$ with the Product Rule and the principle of MATHEMATICAL INDUCTION, which can be stated as follows:

Mathematical induction Suppose P_n is a statement involving the parameter n for each integer $n \geq n_0$, where n_0 is a fixed integer. If (i) P_{n_0} is valid and (ii) P_n implies P_{n+1} for every integer $n \geq n_0$, then P_n is valid for all integers $n \geq n_0$.

This statement of Mathematical Induction looks complicated but actually expresses some very simple logic: if P_{n_0} is valid and P_n implies P_{n+1} for every integer $n \geq n_0$, then P_{n_0+1} is valid. This implies that P_{n_0+2} is valid, which implies that P_{n_0+3} is valid, which implies that P_{n_0+4} is valid, and so forth. Continuing this reasoning indefinitely shows that P_n is valid for all $n \geq n_0$.

To establish (8) for all positive integers n , we set $n_0 = 1$ and let P_n be statement (8). We need to show that (i) P_1 is valid, and that (ii) P_n implies P_{n+1} for all $n \geq 1$.

P_1 is valid because the graph of $y = x^1$ is the line $y = x$ with slope 1.

If P_n is valid for any $n \geq 1$, then by the Product Rule, statement P_1 , and statement P_n , we have

$$\begin{aligned}\frac{d}{dx}(x^{n+1}) &= \frac{d}{dx}(x \cdot x^n) = x \frac{d}{dx}(x^n) + x^n \frac{d}{dx}(x) \\ &= x(nx^{n-1}) + x^n(1) = (n+1)x^n.\end{aligned}$$

Since this is statement P_{n+1} , we have shown that P_n implies P_{n+1} for any $n \geq 1$. This establishes (8) for all integers $n \geq 1$ by mathematical induction.

Interactive Examples 2.6

Interactive solutions are on the web page <http://www.math.ucsd.edu/~ashenk/>.[†]

- Find the derivative $\frac{d}{dx}[(4x^2 + 5)(6x^{-2} - 3)]$. Do not simplify your answer.
- What is $y'(2)$ for $y = \frac{3 + 4/x}{1 + 8/x}$?
- Find $R'(1)$ where $R(s) = \frac{P(s)}{Q(s)}$, $P(1) = 13$, $Q(1) = -2$, $P'(1) = 7$, and $Q'(1) = -4$.
- (a) Give an equation of the tangent line to the curve $y = (1+x+x^2)(1-x^{-1}-x^{-2})$ at $x = 1$ and
 (b) Generate the curve with the tangent line in a suitable window on z calculator or computer.
- Figures 7 and 8 show the graphs of the width $w = w(t)$ (meters) and height $h = h(t)$ (meters) of a rectangle as functions of the time t . What is the approximate rate of change of the area of the rectangle with respect to t at $t = 30$?

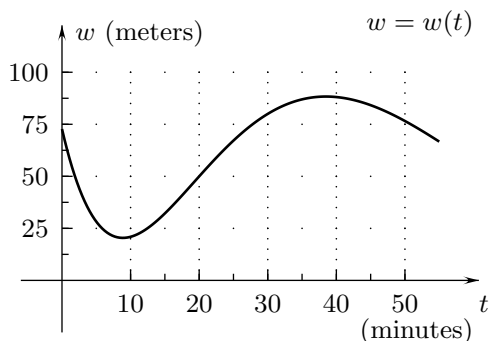


FIGURE 7

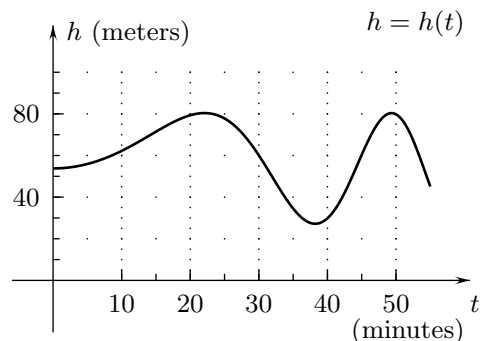


FIGURE 8

[†]In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

Exercises 2.6

^AAnswer provided. ^OOutline of solution provided. ^CGraphing calculator or computer required.

CONCEPTS:

1. (a) Suppose that the width of a rectangle is held constant at $w = 8$ and the height $h = h(t)$ increases, as in Figure 9. Express the rate of change $A'(t)$ of the area in terms of $h'(t)$. (b) Suppose, instead, that the height is held at $h = 6$ and the width $w = w(t)$ is increasing, as in Figure 10. Express $A'(t)$ in terms of $w'(t)$. (c) How are the results of parts (a) and (b) examples of the Product Rule?

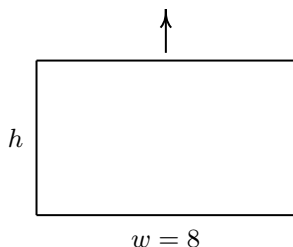


FIGURE 9

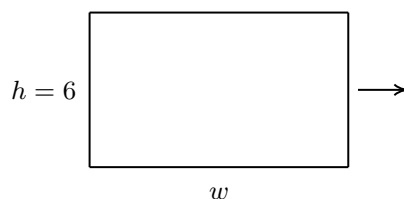


FIGURE 10

2. (a) Use the Product Rule to find a formula for the derivative of $y = f(x)g(x)$, where $f(x) = x^2$ and $g(x) = x^4 + 5$. (b) Find the derivative of $y = f(x)g(x)$ by first simplifying its formula.
3. (a) Use the Quotient Rule to find a formula for the derivative of $y = F(x)/G(x)$, where $F(x) = x^4 + 3$ and $G(x) = x^2$. (b) Find the derivative of $y = F(x)/G(x)$ by first simplifying its formula.

BASICS:

Find the derivatives in Exercises 4 through 7. Do not simplify your answers in Problems 4 or 5.

- 4.^O $\frac{d}{dx}[(5x^3 + 2x^2 - 4)(x^7 - 2x^5)]$
- 5.^O $\frac{dy}{dx}$ for $y = \frac{\sqrt{x}}{x^5 + 1}$
- 6.^O $P'(5)$ where $P(x) = R(x)S(x)$, $R(5) = 3$, $S(5) = 4$, $R'(5) = -3$, and $S'(5) = 10$
- 7.^O $W'(4)$ where $W(x) = \frac{Y(x)}{Z(x)}$, $Y(4) = 2$, $Z(4) = 5$, $Y'(4) = 3$, and $Z'(4) = 6$
- 8.^O Give an equation of the tangent line to $y = (x^2 + x^3 + x^4)(x^5 + x^6 + x^7)$ at $x = 1$.
- 9.^O Figures 11 and 12 give the graphs of differentiable functions $y = A(x)$ and $y = B(x)$. Give approximate values of AB , A/B , and of their first derivatives at $x = 2$.

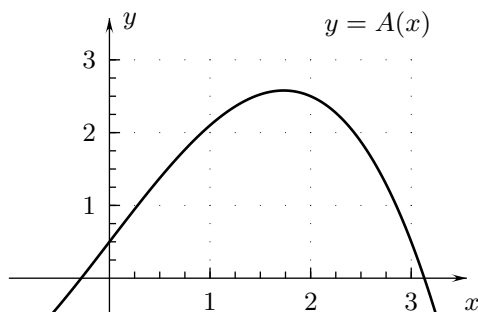


FIGURE 11

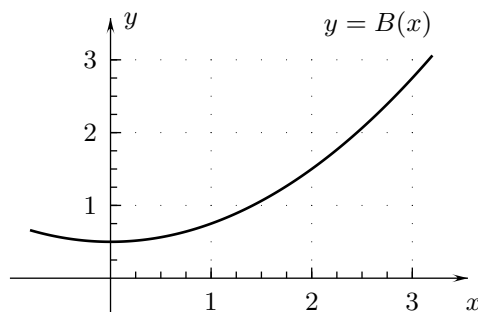


FIGURE 12

10.⁰ Find $\frac{dy}{dx}$ for $y = \frac{ax^2 + 2}{bx^4 + 4}$ with constants a and b . (Do not simplify the answer.)

Find the derivatives in Exercises 11 through 29. ^C In Exercises 23 through 25 check the answers with a procedure of approximate differentiation on a graphing calculator or computer.

11.⁰ $\frac{d}{dx}[(1 + 3x - x^2)(x^2 - 5)]$

18.^A $\frac{d}{dx}\left(\frac{x^3 + 1}{1 - x^2}\right)$

12.^A $\frac{d}{dx}[(x^2 + x + 1)(x^3 - x^2 - x)]$

19. $\frac{d}{dx}\left(\frac{2x + 3}{4x + 5}\right)$

13. $\frac{d}{dx}[(x^2 - 6)(x^2 + 4)]$

20.⁰ $\frac{d}{du}\left(\frac{u^4 + 3}{u^2 + 1}\right)$

14.⁰ $\frac{d}{dx}[(x - x^2)(1 + x^{-1} + x^{-2})]$

21.^A $\frac{d}{dz}\left(\frac{z}{11 - 2z}\right)$

15.^A $\frac{d}{dx}[(x^2 + 5x - 1)(x^{-3} + 2)]$

22. $\frac{d}{dx}\left(\frac{1}{1 + x + x^2}\right)$

16. $\frac{d}{dx}[(x^2 + x^{-2})(2 + 3x)]$

^C **23.⁰** $y'(2)$ for $y = \frac{\sqrt{x}}{1 - \sqrt{x}}$

17.⁰ $\frac{d}{dx}\left(\frac{x^2 + x}{x^2 - 2}\right)$

^C **24.^A** $y'(0)$ for $y = (5 + 3x + 7x^2)(x^3 - 2x + 5)$

^C **25.** $\left.\frac{dy}{dx}\right|_{x=1}$ for $y = \frac{1 + x + x^2}{3 - x^3}$

26.⁰ $f'(10)$ where $f(x) = g(x)h(x)$, $g(10) = -4$, $h(10) = 560$, $g'(10) = 0$, and $h'(10) = 35$

27.⁰ $Z'(1)$ where $Z(s) = \frac{X(s)}{Y(s)}$, $X(1) = 4$, $Y(1) = -10$, $X'(1) = 3$, and $Y'(1) = -4$

28.^A $y'(5)$ where $y(x) = (x^2 + 4)z(x)$, $z(5) = -4$, and $z'(5) = 10$

29. $y'(-3)$ where $y(x) = \frac{z(x)}{1 + x^2}$, $z(-3) = 6$, and $z'(-3) = 15$

In Exercises 30 through 34 (a) give equations of the tangent lines to the graphs at the given values of x . ^C(b) Generate the curves with the tangent lines in suitable windows and copy the drawings on your paper.

30.⁰ $y = (1 + x - x^4)(1 - x + x^3)$ at $x = 1$ **32.** $y = (x^2 - 5x + 1)(x^3 - 5x^2 + 2)$ at $x = 5$

31.^A $y = (1 + x + x^2)(1 - x^{-1} - x^{-2})$ at $x = 1$ **33.⁰** $y = \frac{x^2 - 1}{x^2 + 1}$ at $x = 3$

34. $y = \frac{x^3}{1 - x^2}$ at $x = 2$

35.⁰ At the beginning of 1991 there were 2.1 million farms in the United States with an average size of 467 acres per farm; the number of farms was decreasing 0.035 million farms per year; and the average size was increasing 7 acres per farm per year.⁽³⁾ What was the total acreage of farms and at what rate was it increasing or decreasing at the beginning of 1991?

⁽³⁾Data adapted from *1993 Statistical Abstract of the United States*, Washington, DC: U. S. Bureau of the Census, 1993, p. 655.

- 36.^A** At what rate is the area of a rectangle increasing or decreasing at a moment when it is 5 meters wide and 7 meters high, its width is increasing 3 meters per second, and its height is decreasing 6 meters per second?
- 37.** Figure 13 gives the number of inmates $N = N(t)$ in U.S. federal prisons and Figure 14 gives the percent $P = P(t)$ who were incarcerated for drug offences in year t .⁽⁴⁾ **(a)** Approximately how many more federal prisoners were held for drug offences at the beginning of 1993 than at the beginning of 1979? **(b)** What was the approximate rate of change of the number of federal prisoners being held for drug offences at the beginning of 1988?

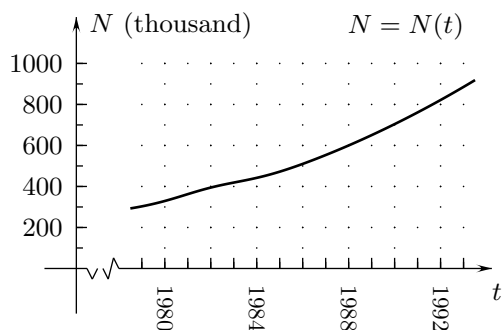


FIGURE 13

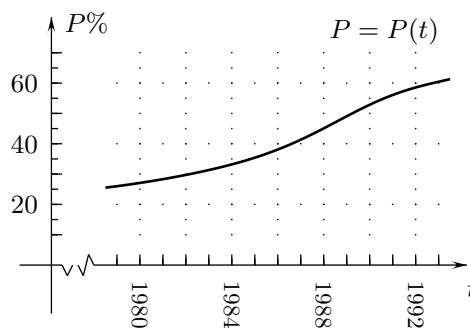


FIGURE 14

- 38.** At the beginning of 1990, annual health care costs in the U.S. were \$2600 per capita and were rising \$260 per capita per year. At that time the population of the U.S. was 250 million and was increasing at the rate of 2.6 million per year.⁽⁵⁾ At what rate were the annual health costs for the entire country increasing at the beginning of 1990?
- 39.^A** What is $P'(3)$ if $P(x) = x^2Q(x)$, $Q(3) = 5$, and $Q'(3) = -6$?
- 40.** What is $Z'(9)$ if $Z(y) = \frac{R(y)}{S(y)}$, $R(9) = 2$, $R'(9) = 4$, $S(9) = 6$, and $S'(9) = 8$?
- 41.** What is $V'(2)$ where $V(x) = \frac{x + U(x)}{x - U(x)}$, $U(2) = 3$, and $U'(2) = -2$?
- 42.^A** What is an object's acceleration in the positive s -direction at $t = 1$ (seconds) if it is at $s = \frac{t + 1}{t + 4}$ (centimeters) on an s -axis at time t ?
- 43.** An object is at $s = (3t^2 + 4)(6t^3 - t + 1)$ (inches) on an s -axis at time t (hours). Give a formula for its acceleration in the positive s -direction as a function of t .
- 44.** What is $\frac{d^2y}{dx^2}$ for $y = (x^2 - 1)(x^3 + x)$?

⁽⁴⁾Data adapted from *Newsweek*, October 17, 1994, New York, NY: Newsweek, Inc., p. 87.

⁽⁵⁾Data adapted from *1993 Statistical Abstract of the United States*, Washington, DC: U. S. Bureau of the Census, 1993, p. 109.

EXPLORATION:

Find the derivatives in Exercises 45 through 52. Simplify your answers.

45.^A dP/dz for $P = (k - \sqrt{z})(k + \sqrt{z})$ with constant k

46. dy/dx for $y = (1 + ax^n)(1 - ax^n)$ with constants a , and n

47.^A $\frac{d}{dt} \left(\frac{\sqrt{t}}{K - t} \right)$ with constant K

48. $\frac{d}{dt} \left(\frac{t}{at^3 + b} \right)$ with constants a and b

49.^A $y'(1)$ for $y = \frac{x^2}{A + Bx}$ with constants A and B

50. Figures 15 and 16 give the number $N = N(t)$ (millions) of MasterCard and Visa accounts and the total outstanding debt $D = D(t)$ (million dollars) in the U.S. as functions of the year.⁽⁶⁾ What were (a) the approximate average debt per credit card and (b) the rate of change with respect to time of the average debt per credit card at the beginning of 1988?

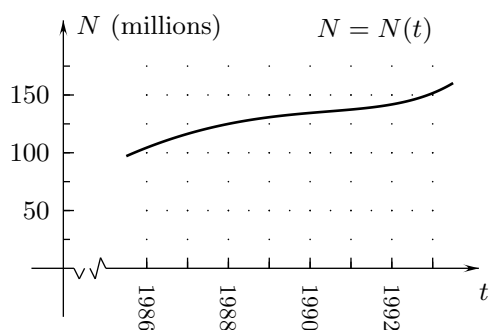


FIGURE 15

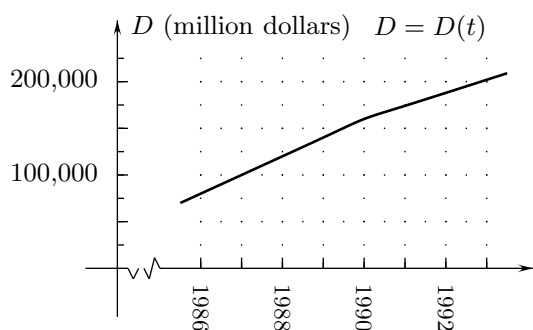


FIGURE 16

51. Figures 17 and 18 give the rate $S = S(t)$ (pairs per year) at which shoes were purchased in the U.S. and the U.S. population $P = P(t)$ (millions) as functions of the year.⁽⁷⁾ What do (a) $\frac{S(t)}{P(t)}$ and (b) $\frac{d}{dt} \left[\frac{S(t)}{P(t)} \right]$ represent, and what are their approximate values at $t = 1982$?

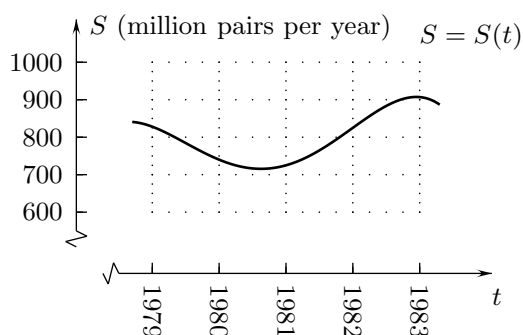


FIGURE 17

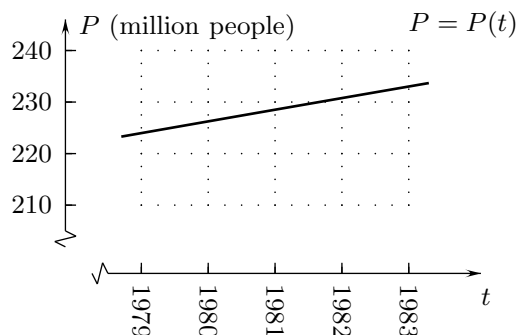


FIGURE 18

⁽⁶⁾Data adapted from *Los Angeles Times*, November 26, 1993, Los Angeles, CA: The Times Mirror Company.

⁽⁷⁾Data adapted from *Technology and Environment* by J. Ausbel and H. Sladovich, Washington, DC: National Academy of Sciences, 1989, p. 53.

- 52.** Imagine that your investments are in the stock market, in real estate, and in livestock. Imagine that on April 15 the value of your stocks is 1.2 million dollars and is rising at the rate of 0.05 million dollars per year; the value of your real estate is 2.1 million dollars and is falling 0.1 million dollars per year; and the value of your livestock is 0.5 million dollars and is rising 0.05 million dollars per year. **(a)** What is the total value of your investments on April 15? **(b)** What percent of your investments is in the stock market on April 15? **(c)** At what rate is the total value of your investments increasing or decreasing on April 15? **(d)** At what rate is the percent of your investments in real estate increasing or decreasing on April 15?
- 53.** What is the derivative of $y = \frac{G(x)}{x}$ at $x = 2$ if the tangent line to $y = G(x)$ at $x = 2$ is $y = 7 + 2x$?
- 54.** What is $W'(2)$ if $W(2) = 6$ and the derivative of $\frac{W(x)}{x}$ is 4 at $x = 2$?

(End of Section 2.6)