Section 2.7

(3/19/08)

Derivatives of powers of functions

OVERVIEW: *In this section we discuss the Chain Rule formula for the derivatives of composite functions that are formed by taking powers of other functions. This is a special case of the general Chain Rule which we will cover in Section 4.1.*

Topics:

- ***The Chain Rule for powers of functions***
- ***On the order of operations***

The Chain Rule for powers of functions

How is the rate of change of the area $A = w^2$ of the square in Figure 1 determined by its width $w = w(t)$ and the rate of change of its width? We can view the area as the product $A = w \cdot w$ and use the Product Rule from the last section to obtain

$$\frac{dA}{dt} = \frac{d}{dt}(w^2) = \frac{d}{dt}(w \cdot w) = w\frac{dw}{dt} + w\frac{dw}{dt} = 2w\frac{dw}{dt}. \tag{1}$$

This shows that the rate of change of the area equals twice the width, multiplied by the rate of change of the width. If we replace $w = w(t)$ by $f = f(x)$ in **(1)**, we obtain $\frac{d}{dx}(f^2) = 2f\frac{df}{dx}$. This is the special case for $n = 2$ of the following general result:

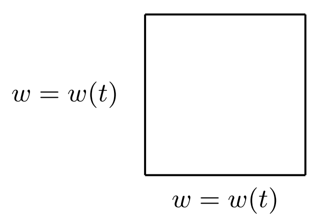


FIGURE 1

Theorem 1 (The Chain Rule for powers of functions) *Suppose that n is a constant and that $y = f(x)$ is a function of x. Then*

$$\frac{d}{dx}(f^n) = nf^{n-1}\,\frac{df}{dx} \tag{2a}$$

or with primes denoting x-derivatives,

$$(f^n)' = nf^{n-1}\,f'. \tag{2b}$$

This formula holds at any x such that $f' = f'(x)$ exists and $f = f(x)$ is in an open interval where $[f(x)]^{n-1}$ is defined.

Remember **(2a)** and **(2b)** as the following statement: the derivative of the nth power of a function equals n, multiplied by the $(n-1)$st power of the function, multiplied by the derivative of the function.

Proof: Suppose that x satisfies the conditions of the theorem. For nonzero Δx, we let Δf denote the change $f(x+\Delta x) - f(x)$ in the value of f from x to $x + \Delta x$, so that $f(x+\Delta x) = f(x) + \Delta f$. Then $\Delta f \to 0$ as $\Delta x \to 0$ since f is continuous at x.

We make the simplifying assumption that $\Delta f \neq 0$ for small, nonzero Δx. Then the derivative of $y = f^n$ at x is the limit as $\Delta x \to 0$ of

$$\frac{[f(x+\Delta x)]^n - [f(x)]^n}{\Delta x} = \frac{(f+\Delta f)^n - f^n}{\Delta x} = \left[\frac{(f+\Delta f)^n - f^n}{\Delta f}\right]\left[\frac{\Delta f}{\Delta x}\right]. \qquad \textbf{(3)}$$

We have written f here for $f(x)$ and we obtained the last expression by multiplying and dividing by Δf.

We let Δx tend to zero. The difference quotient in the second set of brackets on the right of **(3)** tends to the derivative of f with respect to x and the difference quotient in the first set of brackets tends to the derivative of f^n with respect to f. The difference quotient on the left of **(3)** therefore tends to the derivative of f^n with respect to x, and we obtain

$$\frac{d}{df}(f^n) = \frac{d}{df}(f^n)\ \frac{df}{dx}.$$

By Theorem 1 of Section 2.4, $\frac{d}{df}(f^n) = nf^{n-1}$, so the last equation gives

$$\frac{d}{dx}(f^n) = nf^{n-1}\ \frac{df}{dx}$$

which is formula **(2a)** at x. **QED**

Example 1 Find the x-derivative of $y = (x^3+1)^5$.

SOLUTION By **(2a)** with $f(x) = x^3 + 1$ and $n = 5$,

$$\begin{aligned}\frac{d}{dx}[(x^3+1)^5] &= 5(x^3+1)^4 \frac{d}{dx}(x^3+1)\\ &= 5(x^3+1)^4(3x^2) = 15x^2(x^3+1)^4. \ \square\end{aligned}$$

Example 2 What is $z'(0)$ if $z(x) = [y(x)]^4$, $y(0) = 2$ and $y'(0) = -10$?

SOLUTION Formula (2b) with y in place of f and $n = 4$ yields $z' = \frac{d}{dx}(y^4) = 4y^3\, y'$. We set $y = y(0) = 2$ and $y' = y'(0) = -10$ to obtain $z'(0) = 4[y(0)]^3\, y'(0) = 4(2)^3(-10) = -320$. $\square$

Example 3 **(a)** Express the rate of change dV/dt of the volume

$$V = \tfrac{4}{3}\pi r^3 \tag{4}$$

of a sphere in terms of the radius r and the rate of change dr/dt of the radius.
(b) At a particular moment the radius of a sphere is 3 inches and is increasing at the rate of 2 inches per minute. How fast is the volume of the sphere increasing at that moment?

SOLUTION **(a)** By formula **(2a)** with $n = 3$,

$$\frac{dV}{dt} = \frac{d}{dt}(\tfrac{4}{3}\pi r^3) = \tfrac{4}{3}\pi(3r^2)\frac{dr}{dt} = 4\pi r^2\ \frac{dr}{dt}. \tag{5}$$

(b) At the moment under consideration, $r = 3$ and $dr/dt = 5$, so that $dV/dt = 4\pi(3^2)(2) = 72\pi$ cubic inches per minute. □

In the next example, we use formula **(5)** with a graph.

Example 4 Figure 2 shows the graph of the volume $V = V(t)$ of a spherical balloon as a function of the time t. Find **(a)** the approximate radius of the balloon at $t = 6$ and **(b)** the approximate rate of change of the radius with respect to time at $t = 6$.

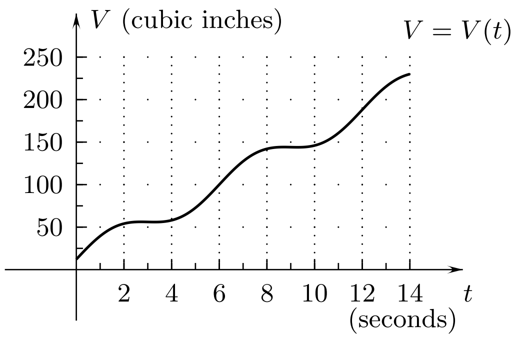


FIGURE 2

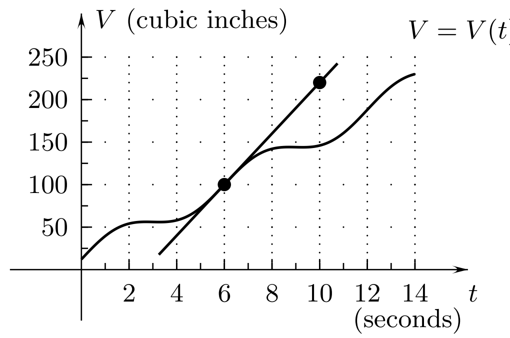


FIGURE 3

SOLUTION **(a)** From the graph in Figure 2 we see that $V(6) \approx 100$ cubic inches, so that by formula **(4)** for the volume of a sphere, $100 \approx \frac{4}{3}\pi r^3$. This gives $r^3 \approx 75/\pi$, so that $r \approx \sqrt[3]{75/\pi} \doteq 2.88$ inches.

(b) To estimate the derivative of the volume, we draw an approximate tangent line to its graph at $t = 6$ as in Figure 3 and find its approximate slope. The points on the line have approximate coordinates $(6, 100)$ and $(10, 220)$. Using these values, we find that the rise ΔV from the first to the second point on the tangent line is approximately $220 - 100 = 120$. The run Δt is 4. Therefore,

$$\left.\frac{dV}{dt}\right|_{t=6} \approx \frac{\Delta V}{\Delta t} \approx \frac{120 \text{ cubic inches}}{4 \text{ seconds}} = 30 \text{ cubic inches per second.}$$

To find dr/dt, we substitute the values $dV/dt \approx 30$ and $r \approx 2.88$ in **(5)**. This yields $30 \approx 4\pi(2.88)^2\dfrac{dr}{dt}$ and then $\dfrac{dr}{dt} \approx \dfrac{30}{4\pi(2.88)^2} \doteq 0.288$ inches per second. □

On the order of operations

Differentiating a complicated expression may require a combination of the Product Rule, the Quotient Rule, the Chain Rule for powers, and other operations. You can determine the order in which to apply these operations by noting the order of the steps used in calculating values of the function. The differentiation is carried out in the reverse order, and this often requires the Product, Quotient, and Chain Rules to differentiate products, quotients, and powers of functions.

Example 5 In what order are the calculations made in finding a value of $f(x) = \left(\dfrac{x+1}{x-3}\right)^5$?

SOLUTION The expressions $y = x + 1$ and $y = x - 3$ are calculated first. Then their quotient is found. And, finally the fifth power is taken. □

Example 6 Find the x-derivative of the function $y = f(x)$ from Example 5.

SOLUTION Because the last step in finding a value of $y = f(x)$ is the taking of the fifth power, we find its derivative by first using the Chain Rule for differentiating the fifth power of a function:

$$\frac{d}{dx}\left[\left(\frac{x+1}{x-3}\right)^5\right] = 5\left(\frac{x+1}{x-3}\right)^4 \frac{d}{dx}\left(\frac{x+1}{x-3}\right).$$

Since the next-to-last step in calculating $f(x)$ involves division, the Quotient Rule is applied next:

$$\begin{aligned}\frac{d}{dx}\left[\left(\frac{x+1}{x-3}\right)^5\right] &= 5\left(\frac{x+1}{x-3}\right)^4 \frac{d}{dx}\left(\frac{x+1}{x-3}\right) \\ &= 5\left(\frac{x+1}{x-3}\right)^4 \left[\frac{(x-3)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x-3)}{(x-3)^2}\right].\end{aligned} \tag{6}$$

Finally, because the first steps in evaluating $f(x)$ are to calculate $y = x+1$ and $y = x-3$, differentiating these functions gives the final result:

$$\begin{aligned}\frac{d}{dx}\left[\left(\frac{x+1}{x-3}\right)^5\right] &= 5\left(\frac{x+1}{x-3}\right)^4 \left[\frac{(x-3)-(x+1)}{(x-3)^2}\right] \\ &= 5\left(\frac{x+1}{x-3}\right)^4 \left[\frac{-4}{(x-3)^2}\right] = \frac{-20(x+1)^4}{(x-3)^6} \ \square\end{aligned} \tag{7}$$

It is a good idea to write down all of the steps, as in equations **(6)** and **(7)**, when you carry out involved differentiations such as in this example so you can concentrate on the details and review your work to see that it is correct.

Example 7 **(a)** Describe the order of operations that are performed to calculate a value of $g(x) = x^2(x^3 + 2x)^{10}$? **(b)** Find the derivative of the function from part (a). Do not simplify the answer.

SOLUTION **(a)** The polynomial $y = x^3 + 2x$ is evaluated and its tenth power is taken; $y = x^2$ is calculated; and then the product is performed.

(b) Because the last step in finding a value of $y = g(x)$ is taking a product, the Product Rule is used first in finding its derivative:

$$g'(x) = \frac{d}{dx}[x^2(x^3 + 2x)^{10}]$$
$$= x^2 \frac{d}{dx}[(x^3 + 2x)^{10}] + (x^3 + 2x)^{10} \frac{d}{dx}(x^2).$$

We use the Chain Rule with the term on the left to obtain

$$g'(x) = x^2[10(x^3 + 2x)^9] \frac{d}{dx}(x^3 + 2x) + (x^3 + 2x)^{10}(2x)$$
$$= 10x^2(x^3 + 2x)^9(3x^2 + 2) + 2x(x^3 + 2x)^{10}.$$

We do not simplify the answer because we have no further use for it. □

Example 8 **(a)** What is the order of operations in evaluating $y = [x^2 + u(x)]^{3/2}$? **(b)** Express the derivative of the function of part (a) in terms of $x, u(x)$, and $u'(x)$.

SOLUTION **(a)** The functions $y = x^2$ and $y = u(x)$ are evaluated, the results are added, and finally the $\frac{3}{2}$ power is taken.

(b) Since the last step in finding a value of $y = [x^2 + u(x)]^{3/2}$ is taking the power, we start with formula **(2)** for differentiating powers. We then differentiate $y = x^2$ and $y = u(x)$:

$$y'(x) = \frac{d}{dx}\{[x^2 + u(x)]^{3/2}\} = \tfrac{3}{2}[x^2 + u(x)]^{1/2} \frac{d}{dx}[x^2 + u(x)]$$
$$= \tfrac{3}{2}[x^2 + u(x)]^{1/2}[2x + u'(x)].\ \square$$

Interactive Examples 2.7

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.[†]

1. Find the derivative dy/dx of $y = (10 - x^{1/2})^{-4}$.

2. What is $W'(0)$ if $W(x) = [W(x)]^4, Z(0) = 2$, and $Z'(0) = -10$?

3. **(a)** Give an equation of the tangent line to $y = (x^2 + 4x)^2$ at $x = -2$. **(b)** Generate the curve and tangent line in a suitable window on a calculator or computer.

4. The volume of punch in a hemispherical bowl of radius 10 inches is $V = 10\pi h^2 \frac{1}{3}\pi h^3$ cubic inches when the punch is h inches deep $(0 \le h \le 10)$. At what rate is the volume increasing at a moment when h is 5 inches and is increasing 6 inches per minute?

[†]In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

Exercises 2.7

[A]Answer provided. [O]Outline of solution provided. [C]Graphing calculator or computer required.

CONCEPTS:

1. Equation **(1)** states that the rate of change of the area of a square equals half its perimeter, multiplied by the rate of change of its width. Why is this plausible? (Imagine that one corner of the square is fixed.)

2. Derive Theorem 1 for $n = 2$ directly from the definition $\dfrac{d}{dx}(f^2) = \lim_{\Delta x \to 0} \dfrac{(f + \Delta f)^2 - f^2}{\Delta x}$, where $f = f(x)$ and $\Delta f = f(x + \Delta x) - f(x)$ on the right. (Expand the square $(f + \Delta f)^2$.)

3. Derive Theorem 1 for $n = -1$ directly from the definition $\dfrac{d}{dx}(f^{-1}) = \lim_{\Delta x \to 0} \dfrac{(f + \Delta f)^{-1} - f^{-1}}{\Delta x}$, where $f = f(x)$ and $\Delta f = f(x + \Delta x) - f(x)$ on the right.

4. How is the formula $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$ for the rate of change of the volume of a sphere from Example 3 related to the area $A = 4\pi r^2$ of the surface of the sphere?

BASICS:

Find the derivatives in Exercises 5 through 19.

5.[O] $\dfrac{d}{dx}[(3x^2 + x)^5]$

6.[O] $\dfrac{d}{dx}\sqrt{3x + 5}$

7.[O] $\dfrac{d}{dx}[(a + bx^2 + cx^5)^{10}]$ with constants a, b, c

8.[O] $G'(3)$ where $G(x) = [y(x)]^{7/2}, y(3) = 1$, and $y'(3) = -5$

History

9.[O] $\dfrac{d}{dx}[(2 + 3x^2)^{10}]$

10.[A] dy/dx for $y = (x + 3x^2)^{-5}$

11. $\dfrac{d}{dt}\sqrt{t^2 - 4}$

12.[O] $G'(z)$ for $G(z) = \sqrt[3]{100z + 45}$

13.[A] $y'(3)$ for $y = (x^4 - 3x^3 + 1)^{100}$

14. $V'(0)$ for $V = (2 - 4t + t^5)^{-3}$

15.[A] $\dfrac{d}{dx}[(3x - 4)^{-1/3}]$

16.[A] $\dfrac{d}{dx}[(6\sqrt{x} + 3)^3]$

17. $\dfrac{d}{dt}[(t^5 - 3t^2 + 1)^{-13}]$

18.[O] $W'(10)$ where $W(u) = [Z(u)]^{1/4}, Z(10) = 16$, and $Z'(10) = -1$

19. $B'(9)$ where $B(v) = [A(v)]^7, A(9) = -1$, and $A'(9) = 12$

In Exercises 20 through 23 **(a)** give equations of the tangent lines at the given values of x. [C]**(b)** Generate the curves and tangent lines in suitable windows and copy them on your paper.

20.[O] The tangent line to $y = (x^4 + 2)^3$ at $x = 1$

21.[O] The tangent line to $y = (x^2 + 4x)^2$ at $x = -2$

22. The tangent line to $y = \sqrt{4 - x}$ at $x = 3$.

23.[A] Figure 4 shows the graph of a differentiable function $y = U(x)$. Find approximate values of **(a)** U, **(b)** $\frac{dU}{dx}$, **(c)** $\frac{d}{dx}(xU)$, and **(d)** $\frac{d}{dx}(U^3)$ at $x = 2$.

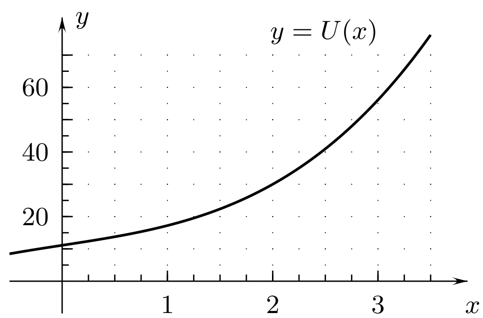


FIGURE 4

24.[O] The density of dry air at a pressure of one atmosphere and a temperature of $T°$C is $\rho(T) = 1.293(1 + 0.00367T)^{-1}$ grams per liter.[(1)] What is the density and the rate of change of the density with respect to temperature at 20°C? Give approximate decimal values.

25.[A] A leaky bucket contains $V(t) = (3 - t)^2$ gallons of water from $t = 0$ (hours) until the bucket is empty. What is the rate of flow out of the bucket when it contains 1 gallon of water?

26. What is the rate of change with respect to time of the volume $V = w^3$ of an expanding cubic crystal at a moment when its width is 10 millimeters and its width is increasing 3 millimetes per day?

27. The one-dimensional density of a rod of mass 160 grams and length L centimeters is $\rho = 160/L$ grams per centimeter. The rod expands when it is heated. What is the rate of change with respect to temperature of its density when it is 40 centimeters long if its length is increasing 0.01 centimeters per degree at that time?

EXPLORATION:

Find the derivatives in Exercises 28 through 31.

28.[O] $\frac{dy}{dx}$ for $y = \frac{(x+1)^4}{2x+3}$.

29.[A] $y'(x)$ for $y = x(5x+4)^{1/4}$.

30.[A] $\frac{d}{dt}(t\sqrt{2t+3})$

31. $y'(t)$ for $y(t) = \frac{(t^3+1)^{10}}{t}$

Give exact and approximate decimal values, as appropriate, in Exercises 32 through 37.

32.[A] The weight of an object is the force of gravity on it. If it weighs 100 pounds on the surface of the earth, then it weighs $w(r) = 100(1 + \frac{1}{4000}r)^{-2}$ pounds at an altitude of r miles above the earth. **(a)** What does the object weigh and **(b)** how rapidly is its weight decreasing when it is 400 miles above the earth if it is rising 15 miles per second? Give exact and approximate decimal values.

33. An object moves along the parabola $y = x^2$ in such a way that its x-coordinate increases at the constant rate of 5 units per minute. **(a)** How rapidly is the object's y-coordinate increasing when its x-coordinate is 1? **(b)** How rapidly is the object's distance to the origin decreasing when its x-coordinate is 1?

34.[A] The force of air resistance (drag) on a car is $D = \frac{1}{30}v^2$ pounds when the velocity is v miles per hour.[(2)] The car is accelerating at a constant rate of 500 miles per hour^2. What is the rate of increase with respect to time of the drag when the car is going 50 miles per hour?

[(1)] *CRC Handbook of Chemistry and Physics*, R. Weast editor, Boca Raton, FL: CRC Press, Inc., 1981, p. F-11.

[(2)] Data adopted from *Fluid Dynamic Drag* by S. Hoerner, Published by the author, 1958, p. 12.

35. Figures 5 and 6 show the graphs of the width $w = w(t)$ (yards) and height $h = h(t)$ (yards) of a rectangular box with a square base as functions of the time t (minutes). What is the approximate rate of change of the volume of the box with respect to t at $t = 10$?

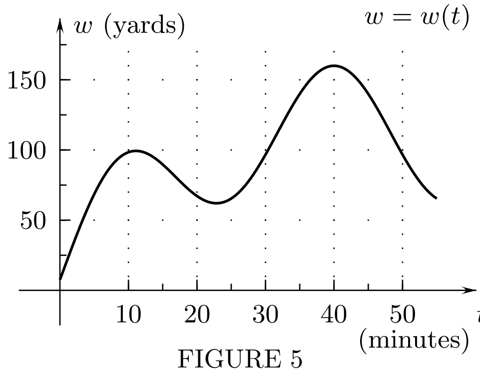


FIGURE 5

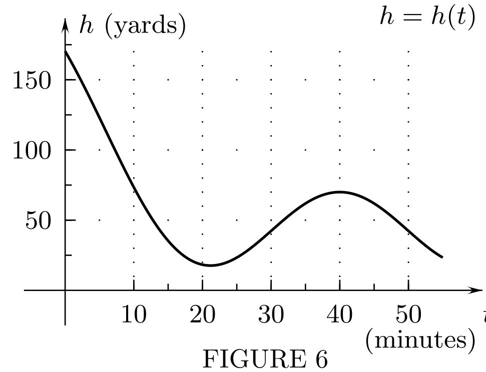
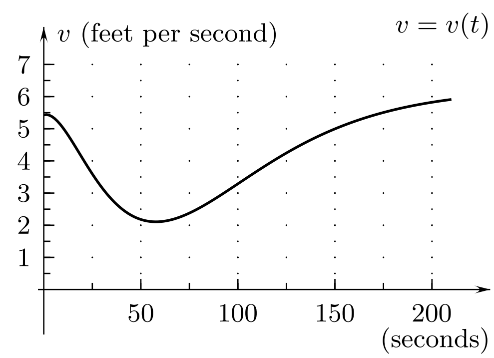


FIGURE 6

36. Imagine a water pipe with a faucet at the end and with the other end connected to a water supply that applies 50 pounds per square inch of water pressure when the faucet is closed. When the valve is opened and water flows through the pipe, the water pressure in the pipe drops. The water pressure in the pipe is $p = 50 - 2v^2$ pounds per square inch when the water is flowing v feet per second. (This is an example of BERNOULLI'S LAW.) Figure 7 shows the graph of the velocity of water as a function of t (seconds) for $0 \le t \le 200$. **(a)** What is the approximate maximum water pressure in the pipe for $0 \le t \le 200$? **(b)** What is the approximate rate of change of the water pressure with respect to t at $t = 150$?

FIGURE 7



37. At what rate is the radius r of a circle decreasing when the area of the circle is 16 square inches if the area is decreasing 3 square inches per minute?

38. The lateral surface area of a right circular cylinder of height h (inches) and base of radius r (inches) is $A = \pi r\sqrt{r^2 + h^2}$. Give a formula for $\frac{dA}{dt}$ in terms of $r, h, \frac{dr}{dt}$, and $\frac{dh}{dt}$.

39. Differentiate $y = (x^3 - 5)^2$ **(a)** by applying the Chain Rule for powers and **(b)** by expanding the square and differentiating the resulting expression.

40. Derive the Quotient Rule from the Product Rule and the Chain Rule for powers by writing $f/g = fg^{-1}$.

(End of Section 2.7)