

INTEGRALS AND APPLICATIONS

The derivative, which we studied in Chapters 2 through 5, is used to find rates of change of functions. In this chapter we begin the study of the second main tool of calculus, the INTEGRAL, which is used to determine changes in values of quantities from their rates of change, to find areas, volumes, weights, average values, lengths of curves, and in many other applications. Section 6.1 gives a preview of the definition of the integral and of the Fundamental Theorem of Calculus in the context of functions whose rates of change are step functions. The DEFINITE INTEGRAL is defined and some of its properties are discussed in Section 6.2/ In Section 6.3 we derive Part I of the FUNDAMENTAL THEOREM OF CALCULUS, which deals with integrals of derivatives. Part II of the Fundamental Theorem concerning derivatives of definite integrals with variable endpoints is discussed in Section 6.4.[†] The Fundamental Theorem is used in Section 6.5 to obtain a formula for definite integrals of power functions $y = x^n$ with constant $n \neq -1$. In this section we also introduce the term INDEFINITE INTEGRAL for “antiderivative.” Section 6.6 deals with finding approximate values of definite integrals of functions given by graphs and tables. Section 6.7 covers integration formulas derived from differentiation formulas for transcendental functions. The technique of INTEGRATION BY SUBSTITUTION is discussed in Section 6.8.

Section 6.1

Step function rates of change

OVERVIEW: *In this section we first look at applications where changes in values of functions can be determined from their rates of change without calculus because the rates of change are step functions. Then we obtain a preview of the definition of definite integrals (Section 6.2) and of Part I of the Fundamental Theorem of Calculus (Section 6.3) by applying the techniques of this section to approximations of continuous rates of change by step functions.*

Topics:

- *Step function rates of change*
- *Approximating continuous rates of change with step functions*

Step function rates of change

We begin with two examples that use the formula for distance traveled at constant velocity,

$$[\text{Distance traveled}] = [\text{Velocity}] \times [\text{Time}].$$

Example 1 The step function $v = v(t)$ of Figure 1 gives a mathematical model of the velocity of a bus. The bus travels 50 miles per hour for two hours, 25 miles per hour for one hour, and 75 miles per hour for three more hours. **(a)** How far does it travel in the entire six hours? **(b)** How is the answer to part (a) related to the area of the rectangles in Figure 2?

[†]Instructors who want to cover Part II of the Fundamental Theorem (derivatives of integrals) before Part I (integrals of derivatives) should cover Section 6.4 before Section 6.3.

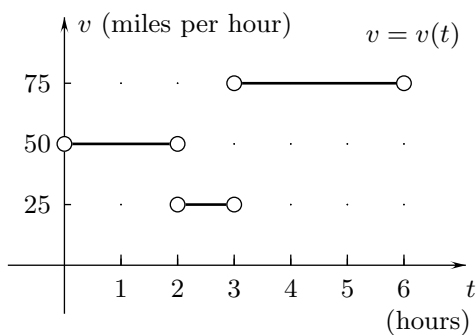


FIGURE 1

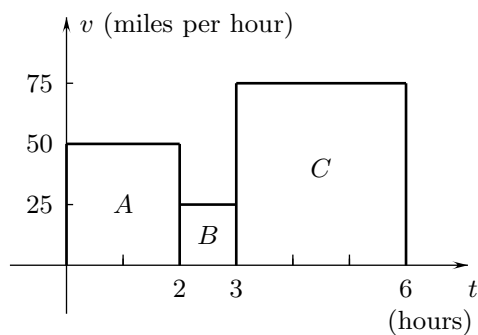


FIGURE 2

SOLUTION (a) The bus travels

$$\begin{aligned} & \left[50 \frac{\text{miles}}{\text{hour}} \right] [2 \text{ hours}] + \left[25 \frac{\text{miles}}{\text{hour}} \right] [1 \text{ hour}] + \left[75 \frac{\text{miles}}{\text{hour}} \right] [3 \text{ hours}] \\ & = 100 + 25 + 225 = 350 \text{ miles} \end{aligned}$$

(b) The distance equals the sum of the areas of rectangles A , B , and C in Figure 2. \square

Example 2

At 10:00 AM one morning a truck driver is 100 miles east of his home town. He drives 75 miles per hour toward the east for two hours to make a delivery. Next, he drives west at 50 miles per hour for two hours to make another delivery and then drives east at 50 miles per hour for two more hours. According to this mathematical model, his velocity toward the east is the step function of Figure 3 with $t = 0$ at 10 AM. (a) How far is he from his home town at $t = 6$? (b) How is the answer to part (a) related to the areas of the rectangles in Figure 4?

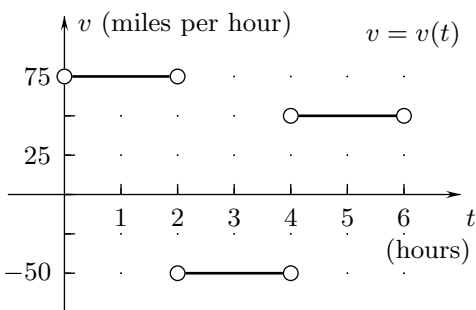


FIGURE 3

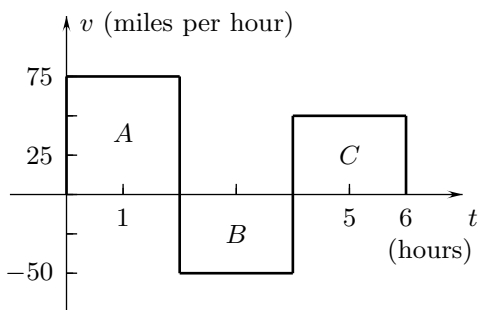


FIGURE 4

SOLUTION (a) At $t = 6$, the driver is

$$\begin{aligned} & [100 \text{ miles}] + \left[75 \frac{\text{miles}}{\text{hour}} \right] [2 \text{ hours}] - \left[50 \frac{\text{miles}}{\text{hour}} \right] [2 \text{ hours}] + \left[50 \frac{\text{miles}}{\text{hour}} \right] [2 \text{ hours}] \\ & = 100 + 150 - 50 + 50 = 250 \text{ miles} \end{aligned}$$

east of his home town.

(b) His location at $t = 6$ equals his location at $t = 0$ plus the the sum of the areas of rectangles A and C in Figure 2, minus the area of rectangle B . \square

The velocity functions in Examples 1 and 2 are step functions. To describe step functions in general, we need the concept of a partition of an interval .

A PARTITION of a finite closed interval $[a, b]$ is a finite number of points $x_0, x_1, x_2, x_3, \dots, x_N$ such that

$$a = x_0 < x_1 < x_2 < x_3 \cdots < x_N = b. \tag{1}$$

These points divide $[a, b]$ into N subintervals. Figure 5 shows, for example, the four subintervals $[x_0, x_1], [x_1, x_2], [x_2, x_3]$, and $[x_3, x_4]$ that are defined by a partition

$$a = x_0 < x_1 < x_2 < x_3 < x_4 = b.$$

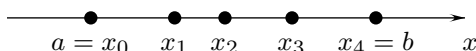


FIGURE 5

Definition 1 (Step functions) A function is a step function on an interval $[a, b]$ if it is constant on the interiors $(x_0, x_1), (x_1, x_2), (x_2, x_3), \dots, (x_{N-1}, x_N)$ of the subintervals in a partition (1) of the interval.

In this section we will deal with step functions, like the function $r = F'(x)$ in Figure 6, that are derivatives of continuous functions.

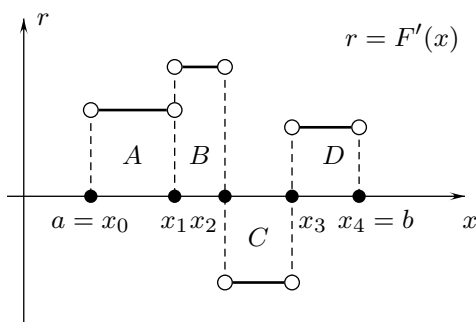


FIGURE 6

The reasoning in the solutions of Examples 1 and 2 leads to the following theorem.

Theorem 1 Suppose that a function $y = F(x)$ is continuous on a finite closed interval $[a, b]$ and that its derivative $r = F'(x)$ is a step function on $[a, b]$. Then the region between the graph $r = F'(x)$ and the x -axis for $a \leq x \leq b$ consists of a finite number of rectangles, and the change in the function's value from $x = a$ to $x = b$ is given by

$$F(b) - F(a) = \left[\begin{array}{c} \text{The area of} \\ \text{all rectangles} \\ \text{above the } x\text{-axis} \end{array} \right] - \left[\begin{array}{c} \text{The area of} \\ \text{all rectangles} \\ \text{below the } x\text{-axis} \end{array} \right]. \tag{2}$$

In the case of the function $r = F'(x)$ of Figure 6, the change $F(b) - F(a)$ in the function is equal to the areas of rectangles A, B, and D above the x -axis minus the area of rectangle C below the x -axis.

Example 3 Use Theorem 1 to solve Example 1.

SOLUTION

We let $s = s(t)$ be the distance the bus of Example 1 travels in t hours. Then $v(t) = s'(t)$ is the positive step function of Figure 1. By Theorem 1 with x replaced by t and $f(x)$ replaced by $s(t)$, $s(6) = s(6) - s(0)$ equals the area $100 + 25 + 225 = 350$ of the three rectangles above the t -axis in Figure 2, as we saw in Example 1. \square

Example 4 Find the answer to Example 2 by applying Theorem 1.

SOLUTION

Let $s = s(t)$ be the truck's distance east of the driver's home town at time t . In this case $v(t) = s'(t)$ is the step function of Figure 3 and by Theorem 1, $s(6) - s(0)$ equals the sum of the areas of rectangles A and C above the t -axis, minus the area of rectangle B below the t -axis in Figure 4. Consequently, $s(6) - s(0) = 150 + 100 - 100 = 150$, and this gives $s(6) = s(0) + 150 = 50 + 150 = 200$ miles, as in Example 2. \square

Example 5 Suppose a water tank contains 300 gallons of water at time $t = 0$ (minutes) and that the rate of flow $r = r(t)$ (gallons per minute) into the tank for $0 \leq t \leq 70$ is the step function in Figure 7. How much water is in the tank at $t = 70$?

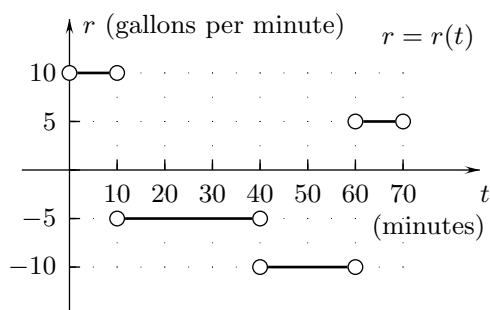


FIGURE 7

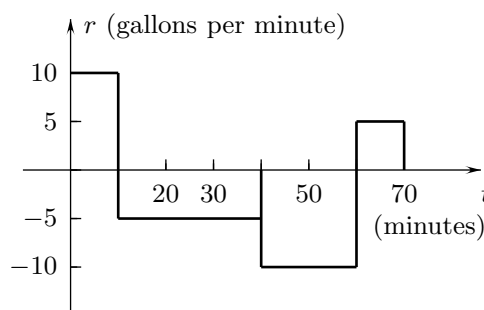


FIGURE 8

SOLUTION

Notice from Figure 7 that the water flows into the tank for $0 < t < 10$ and for $60 < t < 70$, when $r(t)$ is positive and flows out of the tank for $10 < t < 60$, when $r(t)$ is negative.

Let $V = V(t)$ (gallons) be the volume of water in the tank at time t . Then the derivative of V is the step function $r = V'(t)$ of Figure 7. We assume that V is continuous on $[0, 70]$. Then by Theorem 1, $V(70) - V(0)$ equals the area of the first rectangle above the t -axis in Figure 8, minus the area of the two rectangles below the t -axis, plus the area of the last rectangle above the t -axis. We include the units in calculating the areas and write

$$\begin{aligned} V(70) - V(0) &= \left[10 \frac{\text{gallons}}{\text{minute}} \right] [10 \text{ minutes}] - \left[5 \frac{\text{gallons}}{\text{minute}} \right] [30 \text{ minutes}] \\ &\quad - \left[10 \frac{\text{gallons}}{\text{minute}} \right] [20 \text{ minutes}] + \left[5 \frac{\text{gallons}}{\text{minute}} \right] [10 \text{ minutes}] \\ &= 100 - 150 - 200 + 50 \text{ gallons} = -200 \text{ gallons.} \end{aligned}$$

We rewrite this equation in the form,

$$V(70) = V(0) - 200.$$

Since $V(0) = 300$, this gives $V(70) = V(0) - 200 = 300 - 200 = 100$. At $t = 70$ there are 100 gallons of water in the tank. \square

Approximating continuous rates of change with step functions

Suppose that the continuous function $v = v(t)$ of Figure 9 is the velocity of a car that is at $s = s(t)$ on an s -axis at time t , so that $v(t) = s'(t)$. Notice that the region between the graph of the velocity and the t axis for $a \leq t \leq b$ in Figure 9 is in two parts. The region labeled A for the time period $a \leq t < c$, when the car's velocity is positive, is above the t -axis and below the graph. The region labeled B for $c < t \leq b$, when the car's velocity is negative, is below the t -axis and above the graph.

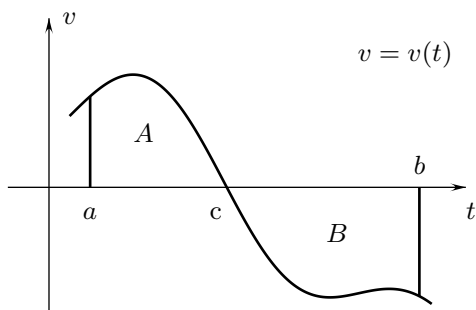


FIGURE 9

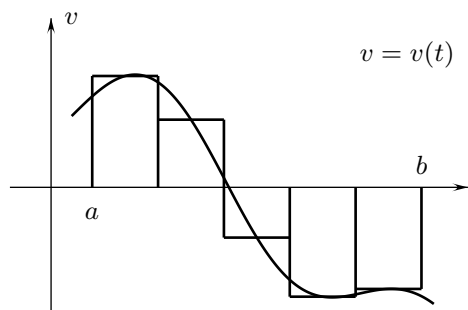


FIGURE 10

We cannot apply Theorem 1 in this case because the velocity is not a step function. Instead we approximate $v = v(t)$ with a step function by approximating regions A and B by rectangles, as in Figure 10, where the sides of the rectangles are determined by a partition of $[a, b]$ and the tops are chosen to intersect the graph of $v = v(t)$ at their midpoints. If the car's velocity were given by the step function, we could apply Theorem 1 and conclude that the change in its position $s(b) - s(a)$ from $t = a$ to $t = b$ equals the area of the two rectangles above the t -axis, minus the area of the three rectangles below the t -axis. Instead, the step function approximates the actual velocity v , and the difference of the areas of the rectangles gives an approximation of the change in the car's position:

$$s(b) - s(a) \approx \left[\begin{array}{c} \text{The area of} \\ \text{the rectangles} \\ \text{above the } t\text{-axis} \end{array} \right] - \left[\begin{array}{c} \text{The area of} \\ \text{the rectangles} \\ \text{below the } t\text{-axis} \end{array} \right]. \quad (3)$$

We could improve on approximation **(3)** of $s(b) - s(a)$ by using more, narrower rectangles, as in Figure 11. Moreover, we could expect to obtain the exact value of $s(b) - s(a)$ by letting the number of rectangles used tend to infinity and their widths tend to zero. We could also expect that the areas of the two sets of rectangles determined by the graph of the continuous function $v = v(t)$ would approach the areas of regions A and B in Figure 9. Consequently, we can expect that

$$\begin{aligned} s(b) - s(a) &= \lim \left\{ \left[\begin{array}{c} \text{The area of} \\ \text{all rectangles} \\ \text{above the } t\text{-axis} \end{array} \right] - \left[\begin{array}{c} \text{The area of} \\ \text{the rectangles} \\ \text{below the } t\text{-axis} \end{array} \right] \right\} \\ &= \left[\begin{array}{c} \text{The area of region} \\ A \text{ in Figure 9} \end{array} \right] - \left[\begin{array}{c} \text{The area of region} \\ B \text{ in Figure 9} \end{array} \right] \end{aligned} \quad (4)$$

where the limit is taken as the number of rectangles used in the approximation tends to ∞ and their widths tend to 0.

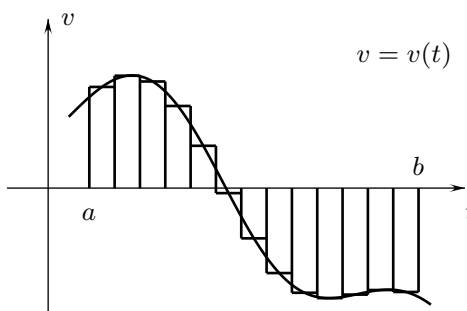


FIGURE 11

In Section 6.2 we will define the INTEGRAL $\int_a^b v(t) dt$ by a limit as in **(4)**. Then, in Section 6.3, we will see that with this definition, equation **(4)** is an example of Part I of the FUNDAMENTAL THEOREM OF CALCULUS.

Interactive Examples 6.1

Interactive solutions are on the web page <http://www.math.ucsd.edu/~ashenk/>.[†]

1. (a) A man runs with the constant velocity of 200 yards per minute for three minutes and then with the constant velocity of 300 yards per minute for five minutes. How far does he run? (b) Draw the graph of the velocity and give a geometric interpretation of the result in part (a).
2. (a) A kite that is initially 250 feet above the ground is allowed to rise at the constant rate of 100 feet per minute for two minutes. Then it is pulled down for three minutes with its height decreasing at the constant rate of 50 feet per minute. Where is the kite at that time? (b) Draw the graph of the kite's upward velocity and use it to give a geometric interpretation of the result of part (a).

[†]In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

Exercises 6.1

^AAnswer provided. ^OOutline of solution provided. ^CGraphing calculator or computer required.

CONCEPTS:

- Suppose that $y = f(x)$ is a continuous function on $[0, 10]$ such that its derivative is a step function and $f(0) = f(10)$. What can you say about the graph of $r = f'(x)$?
- Based on Theorem 1, how are $f(a)$ and $f(b)$ related if f is continuous on $[a, b]$ and $f'(x) = 0$ for $a < x < b$?

BASICS:

- ^O What is $H(10)$ if $y = H(x)$ is continuous on $[2, 10]$, $H(2) = 50$, and $r = H'(x)$ is a step function with $H'(x) = -10$ for $2 < x < 7$, and $H'(x) = 20$ for $7 < x < 10$?
- ^A At 9:00 AM a runner has completed 8 miles of his morning run. He then runs with the constant velocity of 6 miles per hour between 9:00 AM and 9:30 AM and with the constant velocity of 8 miles per hour between 9:30 AM and 10:15 AM. How far has he run by 10:15 AM?
- ^O The step function $v = v(t)$ of Figure 12 gives the velocity in the positive direction at time t of an object as it moves on an s -axis. The scale on the s -axis is given in feet, t is measured in minutes, and v is measured in feet per minute. Suppose that the object's position function is continuous on $[0, 60]$ and that object is $s = 300$ at $t = 10$. Where is it at $t = 60$?

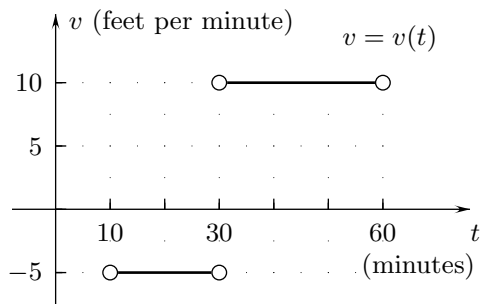


FIGURE 12

- ^O The function $y = F(x)$ is continuous on $[0, 5]$ and its derivative is the step function

$$F'(x) = \begin{cases} 4 & \text{for } 0 \leq x < 2 \\ 7 & \text{for } 2 < x < 3 \\ -3 & \text{for } 3 < x < 5. \end{cases}$$

What is $F(5)$ if $F(0) = 12$?

- ^O What is $F(5)$ if $y = F(x)$ is continuous on $[0, 5]$, $F(0) = -5$, and $r = F'(x)$ is a step function with $F'(x) = 3$ for $0 < x < 3$, and $F'(x) = 7$ for $3 < x < 5$?
- ^A What is $K(100) - K(50)$ if $y = K(x)$ is continuous on $[50, 100]$ and $r = K'(x)$ is a step function such that $K'(x) = 4$ for $50 < x < 70$, $K'(x) = 0$ for $70 < x < 75$, and $K'(x) = 3$ for $75 < x < 100$?
- What is $F(1)$ if $y = F(x)$ is continuous on $[0, 1]$, $F(0) = 0$, and $F'(x)$ equals 10 for $0 < x < \frac{1}{3}$, equals 20 for $\frac{1}{3} < x < \frac{2}{3}$, and equals 30 for $\frac{2}{3} < x < 1$?

- 10.^A The step function of Figure 13 is a mathematical model of the rate of rainfall $r = r(t)$ (inches per year) in Los Angeles from the beginning of 1881 to the beginning of 1886.⁽¹⁾ What was the total rainfall in Los Angeles in the years 1881 through 1885?

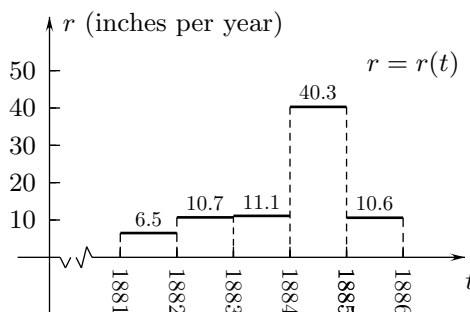


FIGURE 13

11. A few plants, including jack in the pulpit, skunk cabbage, and sacred lotus, can create heat from oxygen and other nutrients. The step function in Figure 14 is a model of the rate of consumption of oxygen by a sacred lotus plant from noon one day to midnight 36 hours later. The flower kept its temperature between 30°C and 37°C as the air temperature varied between 10°C and 35°C by consuming more oxygen in the colder periods.⁽²⁾ How much oxygen did the flower consume during the 36 hours? (t is measured in minutes.)

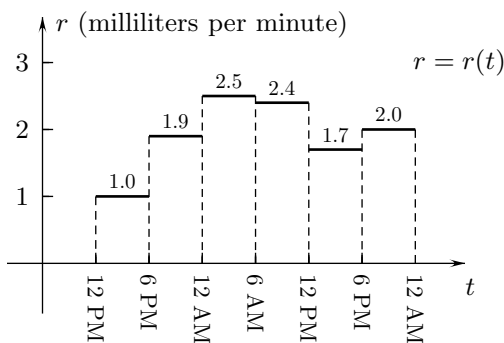


FIGURE 14

12. In a mathematical model of the numbers of manufacturing workers $N = N(t)$ in New York City at time t (years AD), $N(1950) = 740$ (thousand workers) and $N'(t)$ (thousand workers per year) equals -110 for $1950 < t < 1960$, equals -130 for $1960 < t < 1970$, equals -140 for $1970 < t < 1980$, equals -120 for $1980 < t < 1990$, and equals -25 for $1990 < t < 2000$ ⁽³⁾ Based on this data, how many manufacturing workers were there in New York City at the beginning of year 2000?

⁽¹⁾Data adapted from *Los Angeles Times*, January 7, 1993, Source: National Weather Service.

⁽²⁾Data adapted from "Plants that warm themselves" by R. Seymour, *Scientific American*, March, 1997.

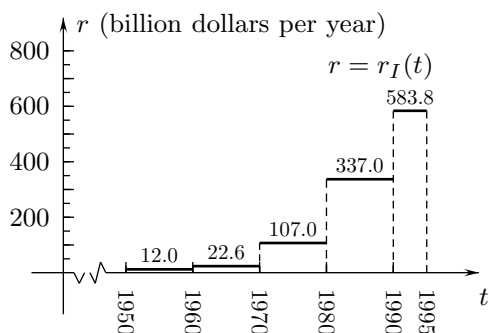
⁽³⁾Data adapted from "Bad Things Happen", *Scientific American*, June, 2001, p.26. Source: US Census Bureau

EXPLORATION:

- 13.^A** The step function $r = r(t)$ defined below gives the approximate rate of U.S. gasoline consumption, measured in millions of barrels per day, from the beginning of 1975 to the beginning of 1995.⁽⁴⁾ Based on this data, how gasoline was consumed in the U.S in the time period $1985 \leq t \leq 1995$ than in the time period $1975 \leq t \leq 1985$? (Disregard leap years and be careful with the units.)

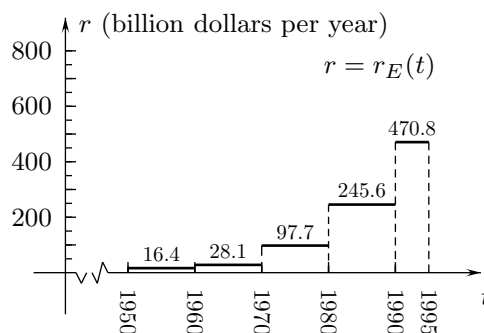
$$r(t) = \begin{cases} 6.2 & \text{for } 1975 \leq t < 1980 \\ 5.8 & \text{for } 1980 \leq t < 1985 \\ 6.3 & \text{for } 1985 \leq t < 1990 \\ 6.5 & \text{for } 1990 \leq t < 1995 \end{cases}$$

- 14.** The functions $r = r_I(t)$ and $r = r_E(t)$ of Figures 15 and 16 give the approximate rates of merchandise imports to and exports from the United States as functions of the year t .⁽⁵⁾ (a) Based on this data, what was the merchandise trade deficit for the time period from 1950 through 1954? (The merchandise trade deficit in a time period is the total value of imports minus the total value of exports during that time.) (b) What was the merchandise trade deficit from 1990 through 1994? (c) What was the ratio of the merchandise trade deficit to the total value of merchandise imports in the period from 1950 through 1954? (d) What was the ratio of the merchandise trade deficit to the total value of merchandise imports in the period from 1990 through 1994?



Merchandise imports

FIGURE 15



Merchandise exports

FIGURE 16

(End of Section 6.1)

⁽⁴⁾Data adapted from "Improving automotive efficiency" by J. DeCicco and M. Ross, *Scientific American*, December, 1994, p. 56.

⁽⁵⁾Data adapted from *Historical Statistics of the United States*, Part 2, Washington, D. C.: U. S. Department of Commerce, p. 886 and *Statistical Abstracts of the United States, 1996*, Washington, D. C.: U. S. Department of Commerce, p. 796.