Section 6.2

(3/20/08)

The definite integral

Overview: *We saw in Section 6.1 how the change of a continuous function over an interval can be calculated from its rate of change if the rate of change is a step function. We also outlined there how in other cases, changes in the function might be determined as limits by using step function approximations of the rates of change. In this section we use this idea in the definition of the* definite integral*. Then we derive some of the basic properties of the integral.*

Topics:

- ***The definite integral***
- ***Piecewise continuous functions***
- ***Integrals and areas***
- ***Special Riemann sums***
- ***Properties of definite integrals***
- ***Unbounded functions***
- ***Riemann sum programs***

The definite integral

The definite integral of the function $y = f(x)$ from $x = a$ to $x = b$ with $a < b$ is a number, denoted $\int_a^b f(x)\,dx$. The symbol $\int$ is called an integral sign, the numbers a and b are the limits of integration, $[a, b]$ is the interval of integration, and $f(x)$ is the integrand.

As we explained at the end of the last section, we want to define the integral so that for the function f of Figure 1, the integral from $x = a$ to $x = b$ equals the area of region A between the graph and the x-axis where $f(x)$ is positive, minus the area of region B between the graph and the x-axis where $f(x)$ is negative. To accomplish this, we approximate $y = f(x)$ by step functions whose graphs form approximations of the two regions by rectangles, as in Figure 2. The integral is defined to be the limit, as the number of rectangles tends to ∞ and their widths tend to zero, of the area of the rectangles above the x-axis, minus the area of the rectangles below the x-axis.

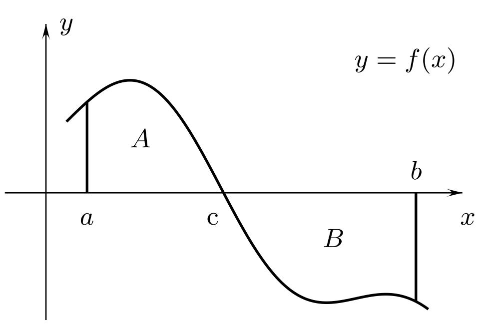


FIGURE 1

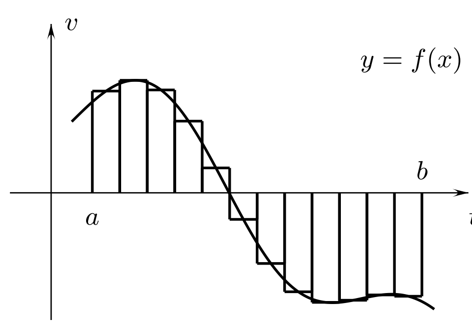


FIGURE 2

To construct approximating rectangles, we start with a partition

$$a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b \tag{1}$$

of the interval $[a, b]$. It divides $[a, b]$ into N subintervals, $[x_0, x_1], [x_1, x_2], \ldots, [x_{N-1}, x_N]$. The jth subinterval is $[x_{j-1}, x_j]$ for $j = 1, 2, 3, \ldots, N$ (Figure 3). We let Δx_j denote its width:

$$[\text{The width of the } j\text{th subinterval}] = \Delta x_j = x_j - x_{j-1}.$$

We also pick, for each j, a point c_j in the jth subinterval that is in the domain of f.

The jth subinterval

$[x_{j-1}, x_j]$

FIGURE 3

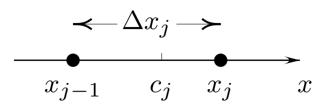


Figure 4 shows the graph of the function f of Figures 1 and 2, and five rectangles that correspond to a partition $a = x_0 < x_1 < x_2 < x_3 < x_4 < x_5 = b$ of $[a, b]$ into five subintervals and to points c_1, c_2, c_3, c_4 and c_5 in the subintervals. For $j = 1$ and 2 on the left, $f(c_j)$ is positive, the base of the rectangle is the jth subinterval on the x-axis, and its top is at $y = f(c_j)$. For $j = 3, 4$, and 5 on the right, $f(c_j)$ is negative, the top of the rectangle is the jth subinterval on the x-axis and its base is at $y = f(c_j)$.

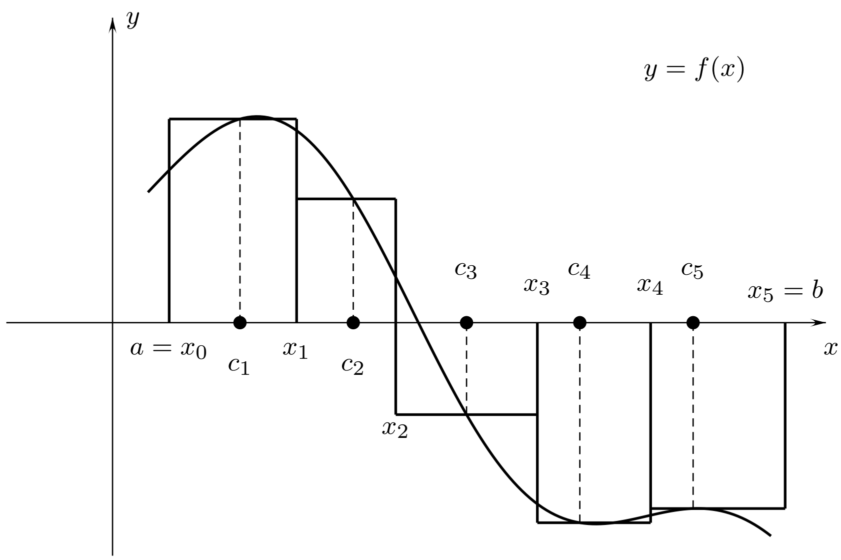
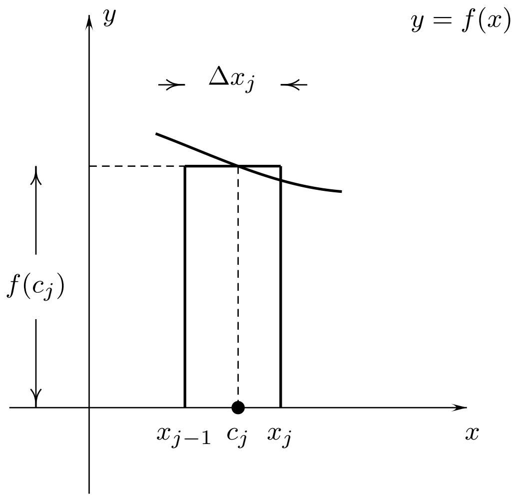


FIGURE 4

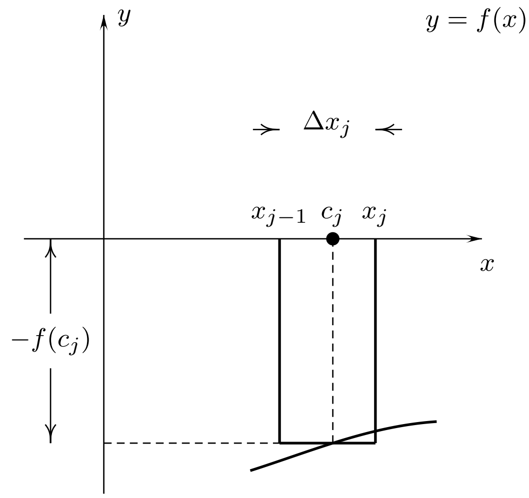
We use the same procedure to construct rectangles for a general partition **(1)**. For each $j = 1, 2, \ldots, N$, we pick a point c_j in the jth subinterval such that $f(c_j)$ is defined. For those values of j for which $f(c_j)$ is positive or zero we construct a rectangle with its bottom formed by the jth subinterval and with top at $y = f(c_j)$ (Figure 5). For those values of j for which $f(c_j)$ negative, we use the jth subinterval as the top of the rectangle and put its bottom at $y = f(c_j)$ (Figure 6). Then the height of the rectangle is $f(c_j)$ if $fc_j)$ is ≥ 0 and is $-f(c_j)$ if $f(c_1) < 0$. In all cases the width of the jth rectangle is the width Δx_j of the subinterval. Consequently,

$$[\text{Area of the } j\text{th rectangle}] = \begin{cases} f(c_j)\Delta x_j \text{ if } f(c_j) \geq 0 \\ -f(c_j)\Delta x_j \text{ if } f(c_j) < 0. \end{cases} \tag{2}$$



$[\text{Area}] = f(c_j)\Delta x_j$

FIGURE 5



$[\text{Area}] = -f(c_j)\Delta x_j$

FIGURE 6

Because of formulas **(2)**, the sum of the areas of the rectangles above the x-axis minus the areas of the rectangles below the x-axis is given by the expression,

$$\begin{aligned}&\left[\begin{matrix}\text{Area of the rectangles}\\ \text{above the } x\text{-axis}\end{matrix}\right] - \left[\begin{matrix}\text{Area of the rectangles}\\ \text{below the } x\text{-axis}\end{matrix}\right]\\ &= f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + \cdots + f(c_N)\Delta x_N \qquad \textbf{(3)}\end{aligned}$$

Notice that no minus are needed in **(3)**: the areas of the rectangles below the x-axis are automatically subtracted because the values $f(c_j)$ are negative for those rectangles.

To express the sum on the right of **(3)** more concisely, we use SUMMATION NOTATION and write

$$\begin{aligned}&\left[\begin{matrix}\text{Area of the rectangles}\\ \text{above the } x\text{-axis}\end{matrix}\right] - \left[\begin{matrix}\text{Area of the rectangles}\\ \text{below the } x\text{-axis}\end{matrix}\right]\\ &= \sum_{j=1}^{N} f(c_j)\Delta x_j \qquad \textbf{(4)}\end{aligned}$$

The symbol $\displaystyle\sum_{j=1}^{N} f(c_j)\Delta x_j$ in **(4)** represents the sum of the quantities $f(c_j)\Delta x_j$ for $j = 1, 2, 3, \ldots, N$ that is given in **(3)**. Summation notation is further illustrated in the following example.

Example 1 **(a)** Write out the sum $\displaystyle\sum_{j=1}^{6} j^2$. (Do not perform the calculations.)

(b) Express $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$ with summation notation.

SOLUTION **(a)** $\displaystyle\sum_{j=1}^{6} j^2$ denotes the sum of the numbers j^2 with $j = 1, 2, 3, 4, 5, 6$. Consequently, it equals $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$.

(b) We write 1 as $\frac{1}{1}$ and let j denote the denominators of the fractions and obtain $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \displaystyle\sum_{j=1}^{8} \frac{1}{j}$. □

The sum **(4)** is called a RIEMANN SUM, and the DEFINITE (or RIEMANN) INTEGRAL of f from a to b, is defined to be a limit of such sums:[†]

Definition 1 (Riemann sums and definite integrals) **(a)** *Consider a function $y = f(x)$ defined at all or at all but a finite number of points in $[a, b]$. A Riemann sum for the integral $\int_a^b f(x)\,dx$ corresponding to a partition $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ of $[a, b]$ is a sum of the form,*

$$\sum_{j=1}^{N} f(c_j)\Delta x_j$$

where for each $j = 1, 2, 3, \ldots, N$, c_j is a point in the jth subinterval where f is defined and $\Delta x_j = x_j - x_{j-1}$ is the width of the jth subinterval.
(b) *The definite integral of f from a to b is the limit of Riemann sums,*

$$\int_a^b f(x)\,dx = \lim \sum_{j=1}^{N} f(c_j)\Delta x_j \qquad \textbf{(5)}$$

as the number N of subintervals in the partitions tends to infinity and their widths tend to zero, provided that the limit exists and is finite.[‡]

Examples of this definition will be examined later in the section.

Piecewise continuous functions

We consider definite integrals of functions which are PIECEWISE CONTINUOUS, according to the following definition.

Definition 2 *The function $y = f(x)$ is* PIECEWISE CONTINUOUS *on $[a, b]$ if it is defined at all but possibly a finite number of points in $[a, b]$ and there is a partition of the interval such that f is continuous on the interior of each subinterval and has finite limits from the right at the left endpoints of the subintervals and finite limits from the left at the right endpoints.*

According to this definition, a piecewise continuous function might not be defined or might be discontinuous at a finite number of other points in the interval.

The function g of Figure 7 is piecewise continuous on the interval $[0, 7]$ because the partition $0 < 3 < 5 < 7$ divides it into three continuous pieces with finite one-sided limits at their endpoints: the function g is defined for all x in $[0, 7]$ except at $x = 3$; it is continuous on the open intervals $(0, 3), (3, 5)$, and $(5, 7)$; and it has finite limits as $x \to 0^+$, as $x \to 3^-$, as $x \to 3^+$, as $x \to 5^-$, as $x \to 5^+$, and as $x \to 7^-$.

[†]Riemann sums and Riemann integrals are named after the German mathematician G. F. B. Riemann (1826–1866), who formulated the modern definition of the integral.

[‡]This type of limit has an $\epsilon\delta$-formulation: the number I is the limit of the Riemann sums if for every positive ϵ, no matter how small, there is a positive δ such that the Riemann sums differ from I by less than ϵ for all partitions into subintervals of widths less than δ.

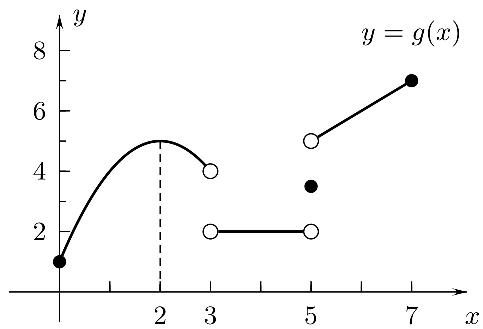


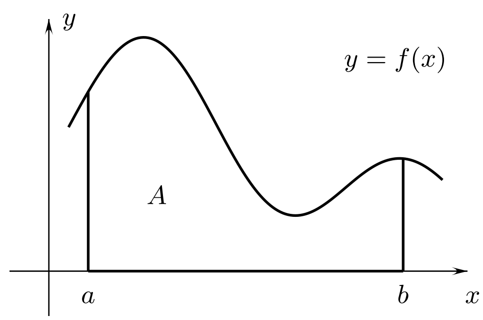
FIGURE 7

The existence of definite integrals of piecewise continuous functions is established by the next theorem, which is proved in advanced courses,

Theorem 1 *The Riemann integral* $\int_a^b f(x)\,dx$ *is defined if* f *is piecewise continuous on the interval* $[a,b]$.

Integrals and areas

Figure 8 shows a region A between the graph of a positive, continuous function f and above the x-axis for $a \le x \le b$. In Definition 1 the integral $\int_a^b f(x)\,dx$ is defined to be the limit of areas of approximations of this region by collections of rectangles such that the approximations become increasingly accurate as the limit is taken. Accordingly, we define the area of A to be the integral:



$$[\text{Area } A] = \int_a^b f(x)\,dx$$

FIGURE 8

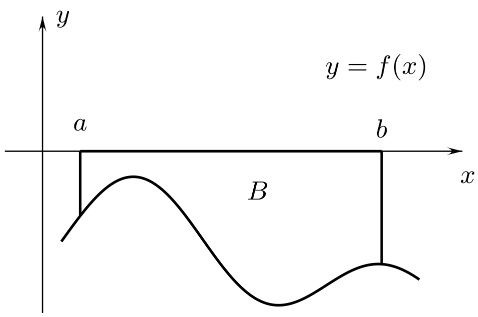
Definition 3 (Areas) *If* f *is piecewise continuous and its values are* ≥ 0 *on* $[a,b]$, *then the area of the region between the graph* $y = f(x)$ *and the* x*-axis for* $a \le x \le b$ *equals the integral* $\int_a^b f(x)\,dx$.

This definition is consistent with other definitions of area if the region consists of rectangles, triangles, or other figures for which there are area formulas from geometry, and it defines the area in other cases.

If f has negative, or positive and negative, values in $[a, b]$, then Riemann sum approximations of $\int_a^b f(x)\,dx$ equal the area of rectangles that approximate the region between the graph and the x-axis where $f(x)$ is positive, minus the area of rectangles that approximate the region between the graph and the x-axis where $f(x)$ is negative. This leads to the following general result relating integrals to areas.

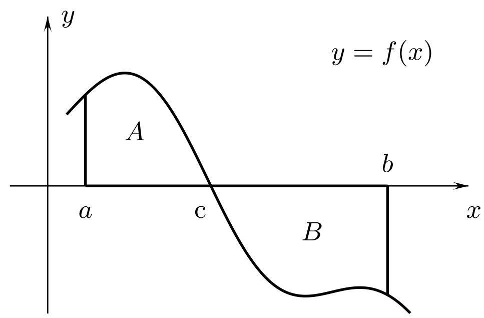
Theorem 2 (Integrals as areas) **(a)** *If f is piecewise continuous and its values are ≤ 0 on $[a, b]$, then the integral $\int_a^b f(x)\,dx$ equals the negative of the area of the region between the graph $y = f(x)$ and the x-axis for $a \leq x \leq b$ (Figure 9).*

(b) *If f is piecewise continuous and has positive and negative values on $[a, b]$, then the integral $\int_a^b f(x)\,dx$ equals the area of the region above the x-axis and below the graph where $f(x)$ is positive, minus the area of the region below the x-axis and above the graph where $f(x)$ is negative. (Figure 10).*



$$\int_a^b f(x)\,dx = -\ [\text{Area } B]$$

FIGURE 9



$$\int_a^b f(x)\,dx = [\text{Area } A] - [\text{Area } B]$$

FIGURE 10

In the next example the value of an integral is found by using Theorem 2 and a formula from geometry.

Example 2 Use the formula for the area of a triangle to evaluate $\int_{-3}^{3}(x+1)\,dx$.

SOLUTION The graph of $y = x + 1$ is the line of slope 1 in Figure 11, which intersects the x-axis at $x = -1$. By Theorem 2, the integral equals the area of triangle B above the x-axis, minus the area of triangle A below the x-axis. Since triangle B is 4 units wide and 4 units high, its area is $\frac{1}{2}(4)(4) = 8$, and since triangle A is 2 units wide and 2 units high, its area is $\frac{1}{2}(2)(2) = 2$. Therefore,

$$\int_{-3}^{3}(x+1)\,dx = [\text{Area } B] - [\text{Area } A] = 8 - 2 = 6. \square$$

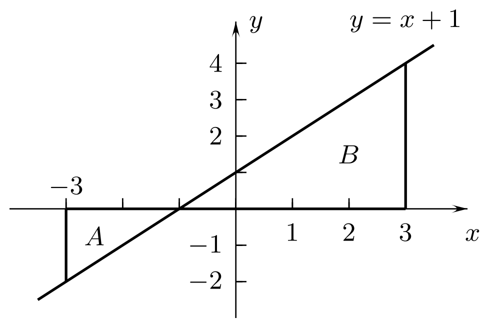


FIGURE 11

Special Riemann sums

A Riemann sum $\sum_{j=1}^{N} f(c_j)\Delta x_j$ for $\int_a^b f(x)\,dx$ is a RIGHT RIEMANN SUM if the points c_j are the right endpoints x_j of the subintervals of the corresponding partition $a = x_0, x_1 < x_2 < \cdots < x_N = b$. It is a LEFT RIEMANN SUM if the c_j's are the left endpoints x_{j-1}, and is a MIDPOINT RIEMANN SUM if the c_j's are the midpoints $\frac{1}{2}(x_{j-1} + x_j)$ of the subintervals.

Example 3 Give **(a)** the right Riemann sum and **(b)** the left Riemann sum for $\int_0^5 (x^3 + 5x)\,dx$ corresponding to the general partition $0 = x_0, < x_1 < x_2 < \cdots < x_N = 5$ of $[0, 5]$.

SOLUTION **(a)** We write, as in Definition 1, $\Delta x_j = x_j - x_{j-1}$ for the width of the jth subinterval. Since x_j is the right endpoint of the jth subinterval, the right Riemann sum is $\sum_{j=1}^{N}[(x_j)^3 - 5x_j]\Delta x_j$.

(b) Becuse x_{j-1} is the left endpoint of the jth subinterval, the left Riemann sum is $\sum_{j=1}^{30}[(x_{j-1})^3 - 5x_{j-1}]\Delta x_j$. $\square$

If the subintervals in a partition for a Riemann sum are of equal width, as in the next example, we write Δx instead of Δx_j for that width.

Example 4 Calculate **(a)** the right Riemann sum, **(b)** the left Riemann sum, and **(c)** the midpoint Riemann sum for $\int_0^1 x^2\,dx$ corresponding to the partition of $[0,1]$ into five equal subintervals. Draw the curve $y = x^2$ with the rectangles whose areas give the Riemann sums.

SOLUTION **(a)** Because $[0,1]$ has width 1, each of the five subintervals in the partition has width $\Delta x = \frac{1}{5} = 0.2$, and the partition is $0 < 0.2 < 0.4 < 0.6 < 0.8 < 1$. The right endpoints are $0.2, 0.4, 0.6, 0.8$, and 1, and the rectangles which give the right Riemann sum touch the curve $y = x^2$ in Figure 12 at their upper right corners. The right Riemann sum is

$$\begin{aligned}&(0.2)^2(0.2) + (0.4)^2(0.2) + (0.6)^2(0.2) + (0.8)^2(0.2) + 1^2(0.2)\\&= [(0.2)^2 + (0.4)^2 + (0.6)^2 + (0.8)^2 + 1^2](0.2)\\&= (0.04 + 0.16 + 0.36 + 0.64 + 1)(0.2) = 0.44.\end{aligned}$$

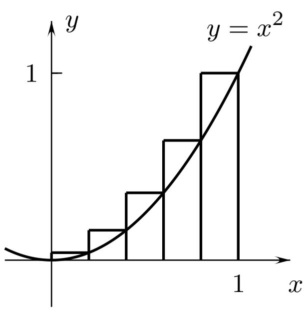


FIGURE 12

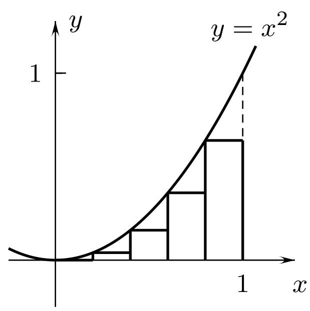


FIGURE 13

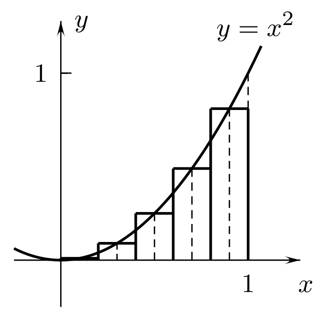


FIGURE 14

(b) The left endpoints of the partition are $0, 0.2, 0.4, 0.6$, and 0.8 and the left Riemann sum equals the area of the rectangles in Figure 13. It equals

$$\begin{aligned}&(0^2)(0.2) + (0.2^2)(0.2) + (0.4^2)(0.2) + (0.6^2)(0.2) + (0.8^2)(0.2))\\&= [(0)^2 + (0.2)^2 + (0.4)^2 + (0.6)^2 + (0.8)^2](0.2)\\&= (0 + 0.04 + 0.16 + 0.36 + 0.64)(0.2) = 0.24.\end{aligned}$$

(c) The midpoint Riemann sum is given by the areas of the rectangles in Figure 14. Since the midpoints of the subintervals are $0.1, 0.3, 0.5, 0.7$, and 0.9, the midpoint Riemann sum is

$$\begin{aligned}&(0.1^2)(0.2) + (0.3^2)(0.2) + (0.5^2)(0.2) + (0.7^2)(0.2) + (0.9^2)(0.2))\\&= [(0.1)^2 + (0.3)^2 + (0.5)^2 + (0.7)^2 + (0.9)^2](0.2)\\&= (0.01 + 0.09 + 0.25 + 0.49 + 0.81)(0.2) = 0.33. \ \Box\end{aligned}$$

A midpoint Riemann sum with equal subintervals, as in Example 4c, is known as a MIDPOINT RULE approximation.

The right Riemann sum (Figure 12) in Example 4 is greater than the left Riemann (Figure 13) bcause $y = x^2$ is increasing for $0 < x < 1$. The midpoint Riemann sum (Figure 14) is the most accurate of the three approximations of the integral.

We will derive all of the formulas for definite integals we will need by using formulas for derivatives and the Fundamental Theorem of Calculus that we discuss in the next section. Some definite integrals can be also calculated directly from Definition 1 and formulas from algebra, as in the next example.

Example 5 The formula

$$1^2 + 2^2 + 3^2 + \cdots + N^2 = \tfrac{1}{3}N^3 + \tfrac{1}{2}N^2 + \tfrac{1}{6}N \qquad \textbf{(6)}$$

for the sum of the square of the first N positive integers is is derived in Exercise 50. Use it and Definition 1 with right Riemann sums and partitions into equal subintervals to find the value of $\int_0^1 x^2\,dx$.

SOLUTION For any positive integer N, the partition of $[0,1]$ into N equal subintervals is

$$0 < \frac{1}{N} < \frac{2}{N} < \frac{3}{N} < \cdots < \frac{N-1}{N} < 1.$$

The width of each subinterval is $\Delta x = 1/N$, and the right endpoint of the jth subinterval is $x_j = j/N$. Therefore, the right Riemann sum for $\int_0^1 x^2\,dx$ with this partition is

$$\sum_{j=1}^{N}(x_j)^2\Delta x = \left[\left(\frac{1}{N}\right)^2 + \left(\frac{2}{N}\right)^2 + \left(\frac{3}{N}\right)^2 + \cdots + \left(\frac{N}{N}\right)^2\right]\left(\frac{1}{N}\right)$$
$$= \frac{1}{N^3}(1^2 + 2^2 + \cdots N^2).$$

With formula **(6)** we obtain

$$\sum_{j=1}^{N}(x_j)^2\Delta x = \frac{1}{N^3}\left(\tfrac{1}{3}N^3 + \tfrac{1}{2}N^2 + \tfrac{1}{6}N\right) = \tfrac{1}{3} + \frac{1}{N} + \frac{1}{6N^2} \qquad \textbf{(7)}$$

Because the widths of the subintervals tend to zero as the number N of subintervals tends to ∞, the integral is the limit of the Riemann sums **(7)** as $N \to \infty$:

$$\int_0^1 x^2\,dx = \lim_{N\to\infty}\sum_{j=1}^{N}(x_j)^2\Delta x = \lim_{N\to\infty}\left(\tfrac{1}{3} + \frac{1}{N} + \frac{1}{6N^2}\right) = \tfrac{1}{3}. \ \square$$

Properties of definite integrals

We now give one definition and two theorems with basic properties of definite integrals.

Because regions of zero width have zero area, integrals with equal limits of integration are defined to be zero. Also, an integral from b to a with $a < b$ is defined to be the negative of the integral from a to b:

Definition 4 **(a)** *For any function f,*

$$\int_a^a f(x)\,dx = 0. \tag{8}$$

(b) *If the integral of f from a to b is defined with $a < b$, then*

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx. \tag{9}$$

Example 6 **(a)** What is the value of $\int_{10}^{10} x^2\,dx$? **(b)** In Example 5 we found that $\int_0^1 x^2\,dx = \frac{1}{3}$. What is the value of $\int_1^0 x^2\,dx$?

SOLUTION **(a)** $\int_{10}^{10} x^2\,dx = 0$ by part (a) of Definition 4.

(b) $\int_1^0 x^2\,dx = -\int_0^1 x^2\,dx = -\frac{1}{3}$ by part (b) of the definition. □

Theorem 3 (Integrals over adjacent intervals) *If f is piecewise continuous on an interval containing the numbers a, b, and c, then*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx. \tag{10}$$

Partial proof: Formula **(10)** holds for $a < b < c$ because a Riemann sum for the integral from a to b plus a Riemann sum from b to c is a Riemann sum for the integral from a to c and the integrals are the limits of the Riemann sums.

Formula **(10)** in other cases follows by applying Definition 4. For instance, if $a < c < b$, then by **(9)** and **(10)**,

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \\ &= \int_a^c f(x)\,dx - \int_b^c f(x)\,dx.\end{aligned}$$

Adding $\int_b^c f(x)\,dx$ to both sides of this equation gives **(10)** in this case. **QED**

Theorem 3 for $a < b < c$ and a positive function f has the geometric interpretation illustrated in Figure 15. The left side of equation **(9)** is equal to the area between the graph and the x-axis for $a \le x \le c$, the integrals on the right are equal to the areas for $a \le x \le b$ and for $b \le x \le c$, and the area of the whole region is equal to the sum of the area of its two parts.

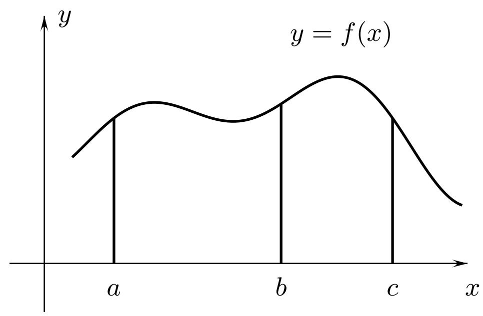


FIGURE 15

Example 7 What is the integral of g from 1 to 8 if its integral from 1 to 5 is -7 and its integral from 5 to 8 is 9?

SOLUTION $\displaystyle\int_1^8 g(x)\,dx = \int_1^5 g(x)\,dx + \int_5^8 g(x)\,dx = -7 + 9 = 2$ □

Theorem 4 (Integrals of linear combinations) *If f and g are piecewise continuous on an interval containg a to b, then for any constants A and B,*

$$\int_a^b [Af(x) + Bg(x)]\,dx = A\int_a^b f(x)\,dx + B\int_a^b g(x)\,dx. \qquad \textbf{(11)}$$

Proof: Any Riemann sum for the integral of $y = Af(x) + Bg(x)$ equals A multiplied by a Riemann sum for the integral of f, plus B multiplied by a Riemann sum for the integral of g:

$$\sum_{j=1}^{N}[Af(c_j) + Bg(c_j)]\Delta x = A\sum_{j=1}^{N} f(c_j)\Delta x + B\sum_{j=1}^{N} g(c_j)\Delta x.$$

Formula **(11)** follows since the integrals are the limits of their respective Riemann sums. **QED**

Example 8 What is $\displaystyle\int_{-35}^{35} [2p(x) - 3q(x)]\,dx$ if $\displaystyle\int_{-35}^{35} p(x)\,dx = 10$ and $\displaystyle\int_{-35}^{35} q(x)\,dx = 20$?

SOLUTION With **(11)** we obtain

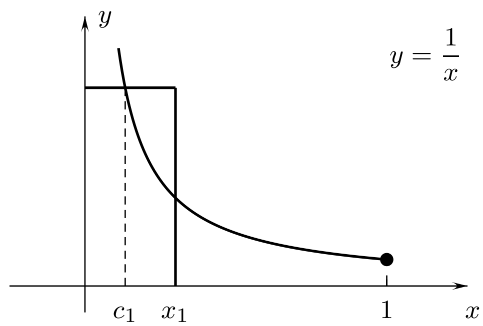
$$\begin{aligned}\int_{-35}^{35} [2p(x) - 3q(x)]\,dx &= 2\int_{-35}^{35} p(x)\,dx - 3\int_{-35}^{35} q(x)\,dx \\ &= 2(10) - 3(20) = -40. \ \square\end{aligned}$$

Unbounded functions

A function f is BOUNDED on an interval $[a, b]$ if there is a constant M such that $|f(x)| \leq M$ for all x in the interval where $f(x)$ is defined. Piecewise continuous functions, which are the type of functions we use in definite integrals, are bounded on finite intervals.[†]

The Riemann integral $\int_a^b f(x)\,dx$ is not defined if f is not bounded on $[a, b]$. For example, $\int_0^1 \frac{1}{x}\,dx$ is not defined as a Riemann integral because $1/x \to \infty$ as $x \to 0^+$ and consequently $y = 1/x$ is not bounded on the interval of integration $[0, 1]$ (Figure 16). The Riemann sums for this integral cannot have a finite limit because (as is illustrated in Exercise 5 below) arbitrarily large Riemann sums can be constructed for any partition by choosing c_1 sufficiently close to 0.

FIGURE 16



We will see in Section 7.7 that integrals of certain unbounded functions can be defined as IMPROPER INTEGRALS. These are limits of Riemann integrals.

C ***Riemann sum programs***

Programs or other procedures are available for many graphing calculators and computer calculus software for calculating Riemann sums. Many of these programs and procedures also generate the graphs of functions with the rectangles whose areas give the Riemann sums.[‡]

C ***Example 9*** **(a)** Use a Riemann sum program or procedure on a calculator or computer to find the midpoint Riemann sum for $\int_0^1 x^3\,dx$ with partitions of $[0, 1]$ into 10, 20, 50, and 100 equal subintervals. Use the window $-0.25 \leq x \leq 1.25$, $-0.25 \leq y \leq 1.25$ if the program or precedure generates graphs. **(b)** Use the results of part (a) to predict the exact value of $\int_0^1 x^3\,dx$.

[†]If f is piecewise continuous on $[a, b]$, then in the interior of each subinterval of a partition $a = x_0 < x_1 < \cdots < x_N = b$, it is equal to a function that is is continuous in the closure of that subinterval. The latter function is bounded on the subinterval by the Extreme Value Theorem, so f is bounded on $[a, b]$.

[‡]Riemann sum programs for Texas Instruments calculators can be found at the web site for the text, www.math.ucsd.edu/~ashenk/.

SOLUTION **(a)** The midpoint Riemann sum is 0.24875 for $N = 10$, is 0.2.496875 for $N = 20$, is 0.24995 for $N = 50$, and is 0.2499875 for $N = 100$. The rectangles for $N = 10$ and 20 are shown in Figures 17 and 18.
(b) It appears that the Riemann sums are approaching $\frac{1}{4}$ as N increases, so we predict that $\int_0^1 x^3\,dx = \frac{1}{4}$. (We could confirm this with integration formulas that we will derive in Section 6.5.) □

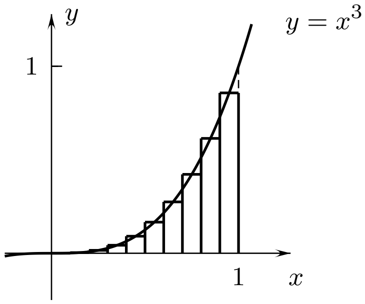


FIGURE 17

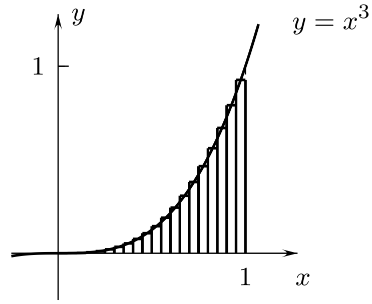


FIGURE 18

Interactive Examples 6.2

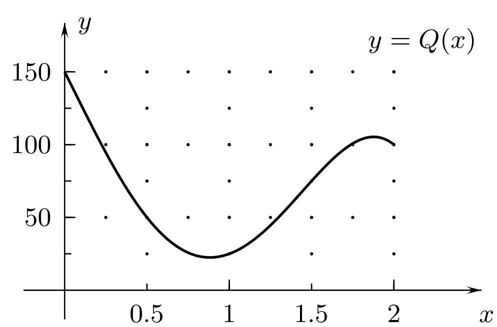
Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.†

1. Use area formulas from basic geometry to find the value of $\int_{-2}^{4} (\frac{1}{2}x - 2)\,dx$.

2. Calculate the left Riemann sum for $\int_1^8 \frac{1}{x}\,dx$ for the partition $1 < 2 < 4 < 8$ of $[1, 8]$. (Notice the unequal subintervals.)

3. Calculate the right Riemann sum for $\int_0^2 Q(x)\,dx$ relative to the partition of $[0, 2]$ into four equal subintervals, where $y = Q(x)$ is the function whose graph is shown in Figure 19.

FIGURE 19



†In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

4. Use three rectangles of equal width with the graph $y = f(x)$ in Figure 20 to find the approximate value of $\int_0^{15} f(x)\,dx$.

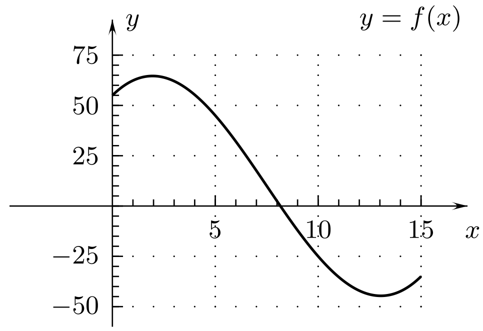


FIGURE 20

C **5.** Use at least three Riemann sums, calculated with a Riemann sum procedure on a calculator or computer to predict the value of $\int_0^4 (15\sqrt{x} - 9x)\,dx$.

6. What is the value of $\int_3^1 F(x)\,dx$ if $\int_1^4 F(x)\,dx = -4$ and $\int_3^4 F(x)\,dx = 6$?

Exercises 6.2

[A]Answer provided. [O]Outline of solution provided. [C]Graphing calculator or computer required.

CONCEPTS:

1. **(a)** Draw the region between the curve $y = 1-x^2$ and the x-axis for $0 \le x \le 1$ and the rectangles that give the left, right, and midpoint Riemann sums for the integral $\int_0^1 (1-x^2)$ relative to the partition $0 < 0.25 < 0.5 < 0.75 < 1$. **(b)** Explain why the left Riemann sum is less than the integral, why the right Riemann sum is greater than the integral, and why of the three sums the midpoint Riemann sum gives the best approximation of the integral.

2. Express the equations **(a)** $\int_0^5 2k\,dx = 2\int_0^5 k\,dx$ and **(b)** $\int_0^{15} k\,dx = 3\int_0^5 k\,dx$ for positive constants k as statements about areas of rectangles.

3. Rewrite the the equation $\int_1^4 f(x)\,dx = \int_1^6 f(x)\,dx + \int_6^4 f(x)\,dx$ for a positive function $y = f(x)$ so that it becomes a statement about areas.

4. Derive **(10)** in the case of $a < c < b$ and a positive continuous f by considering areas.

5. Suppose that $x_1 = \frac{1}{3}$ in Figure 16. Illustrate the fact that Riemann sums for $\int_0^1 \frac{1}{x}\,dx$ can be arbitrarily large by finding values of c_1 such that the rectangle in that drawing **(a)** has area 10, **(b)** has area 100, and **(c)** has area 1000.

BASICS:

6. Use the formula for the area of a triangle to find the value of $\int_0^3 (x-1)\,dx$.

7. Use the fact that the curve $y = \sqrt{16 - x^2}$ is the upper half of the circle $x^2 + y^2 = 16$ of radius 4 with its center at the origin to find the exact value of $\int_{-4}^0 \sqrt{16 - x^2}\,dx$.

8. Use areas to evaluate $\int_1^6 P(x)\,dx$ where $P(x) = \begin{cases} -1 & \text{for} \quad 1 < x < 3 \\ 3 & \text{for} \quad 3 < x < 4 \\ 2 & \text{for} \quad 4 < x < 6. \end{cases}$

9.[O] Calculate $\sum_{j=1}^{6} j(j-1)(j-2)$.

10.[O] Express $1^2 + 2^2 + 3^2 + \cdots + 99^2$ with summation notation.

11. Calculate **(a**[A]**)** $\sum_{j=1}^{5}\left(\frac{j}{12}\right)$, **(b**[A]**)** $\sum_{j=1}^{5}\left(\frac{12}{j}\right)$, **(c)** $\sum_{j=1}^{5}\left(1-\frac{j}{5}\right)$, and **(d)** $\sum_{j=1}^{6}\left[\frac{j}{6}-\left(\frac{j}{6}\right)^2\right]\left(\frac{1}{6}\right)$.

12. Express **(a**[A]**)** $\frac{1}{3}+\frac{2}{4}+\frac{3}{5}+\frac{4}{6}+\cdots+\frac{175}{177}$, **(b)** $5+\frac{5}{4}+\frac{5}{9}+\frac{5}{16}+\cdots+\frac{5}{121}$, and **(c)** $2^3+\left(\frac{2}{2}\right)^3+\left(\frac{2}{3}\right)^3+\left(\frac{2}{4}\right)^3+\cdots+\left(\frac{2}{N}\right)^3$ with summation notation.

13.[O] Calculate the right Riemann sum for $\int_0^2 (4x - x^2)\,dx$ corresponding to the partition $0 < 0.5 < 1 < 1.5 < 2$.

14.[O] Follow the instructions in Exercise 13, but with the left Riemann sum.

15.[O] Calculate the left Riemann sum for $\int_0^9 \sqrt{x}\,dx$ for the partition $0 < 3 < 5 < 9$ of $[0, 9]$. (Notice the unequal subintervals.)

16. What is the right Riemann sum for $\int_0^{10} x^2\,dx$ for the partition $0 < 5 < 9 < 10$ of $[0, 10]$.. (Notice the unequal subintervals.)

17.[A] Calculate **(a)** the left and **(b)** the right Riemann sums for $\int_0^2 P(x)\,dx$ relative to the partition of $[0, 2]$ into four equal subintervals, where $y = P(x)$ is the function whose graph is shown in Figure 21.

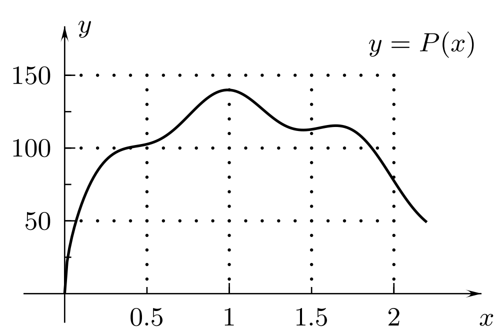


FIGURE 21

18.[A] The graph of $y = R(x)$ is shown in Figure 22. What is the midpoint Riemann sum for $\int_0^{40} R(x)\,dx$ corresponding to the partition of $[0, 40]$ into four equal subintervals?

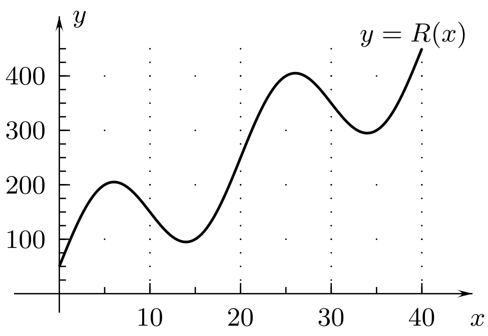


FIGURE 22

C **19.**[O] Predict the value of $\int_0^2 (4x - x^2)\,dx$ by using a Riemann-sum procedure on a calculator or computer to calculate midpoint Riemann sums with 20, 50, and 200 equal subintervals. Use the window $0 \le x \le 2, -1 \le y \le 5$.

Use at least three Riemann sums, calculated with a Riemann sum procedure on a calculator or computer, to predict the values of the integrals in Exercises 20 through 23.

C **20.**[O] $\int_1^4 \frac{1}{x^2}\,dx$

C **21.**[A] $\int_0^1 \pi \sin(\pi x)\,dx$

C **22.** $\int_1^4 \frac{1}{x^2}\,dx$

C **23.** $\int_0^5 (100x - 4x^3)\,dx$

24.[O] What is $\int_{-7}^7 [6f(x) + 3g(x)]\,dx$ if $\int_{-7}^7 f(x)\,dx = 4$ and $\int_{-7}^7 g(x)\,dx = -5$?

25.[O] What is $\int_1^6 h(x)\,dx$ if $\int_1^2 h(x)\,dx = 10$ and $\int_2^6 h(x)\,dx = 7$?

In Exercises 26 through 34 use areas formula from basic geometry to find the values of the integrals. Draw the region(s) whose area(s) give the integral.

26.[O] $\int_{-2}^1 (-2x)\,dx$

27.[A] $\int_{-20}^{40} x\,dx$

28. $\int_{-10}^5 6\,dx$

29. $\int_{-3.96}^{5.72} 0\,dx$

30.[A] $\int_{-1}^0 \sqrt{1 - x^2}\,dx$

31. $\int_{-5}^5 -\sqrt{25 - x^2}\,dx$

32.[O] $\int_3^1 7\,dx$

33. $\int_5^0 x\,dx$

34.[A] $\int_{-2}^3 H(x)\,dx$ where $H(x) = \begin{cases} \sqrt{4 - x^2} & \text{for } -2 \le x \le 0 \\ -\sqrt{9 - x^2} & \text{for } 0 < x \le 3 \end{cases}$

35.[O] What is $\int_{10}^0 Q(x)\,dx$ if $\int_0^{12} Q(x)\,dx = 35$ and $\int_{10}^{12} Q(x)\,dx = 20$?

36.[A] What is $\int_3^0 Y(x)\,dx$ if $\int_0^4 Y(x)\,dx = 100$ and $\int_4^3 Y(x)\,dx = -25$?

37. What is $\int_1^{10} P(x)\,dx$ if $\int_{10}^5 P(x)\,dx = 100$ and $\int_1^5 P(x)\,dx = -25$?

38.[O] What is $\int_0^3 [2f(x) - 4g(x)]\,dx$ if $\int_0^3 f(x)\,dx = 100$ and $\int_0^3 g(x)\,dx = 200$?

39. What is $\int_{-5}^5 [50r(x) + 25s(x)]\,dx$ if $\int_{-5}^5 r(x)\,dx = 2$ and $\int_{-5}^5 s(x)\,dx = 3$?

EXPLORATION:

40.[A] Use the formulas for areas of rectangles and circles to evaluate $\int_0^2 (5 - 3\sqrt{4 - x^2})\,dx$.

41.[O] Give the right Riemann sum for $\int_0^5 (1 + x^2)\,dx$ corresponding to the partition of $[0, 5]$ into N equal subintervals.

42. Give the left Riemann sum for $\int_0^\pi \sin x\,dx$ corresponding to the partition of $[0, \pi]$ into N equal subintervals.

43.[A] Give the general Riemann sum for $\int_{-3}^3 (x + x^5)\,dx$.

44. Give the general Riemann sum for $\int_{-\pi/4}^{\pi/4} \sec x\,dx$.

[C] **45.**[A] The integral $\int_{0.5}^4 \left(8x + \frac{4}{x^2}\right) dx$ has the value 70. Use a Riemann-sum procedure and trial and error to find a positive integer N such that the midpoint Riemann sum with N equal subintervals for the integral differs from the integral by less than 0.1 and more than 0.09.

[C] **46.** Use the Riemann-sum program and trial and error to find a positive integer N such that the left and right Riemann sums with N equal subintervals for $\int_0^4 \sqrt{1 + x^3}\,dx$ differ by more than 1 and less than 1.2.

[C] **47.**[A] How much larger is the right Riemann sum than the left Riemann sum for $\int_0^2 (3x^2 - x^3)\,dx$ **(a)** with the partition into four equal subintervals, **(b)** with the partition of $[0, 2]$ into 20 equal subintervals, and **(c)** with the partition of $[0, 2]$ into 80 equal subintervals?

[C] **48.** **(a)** Find the value of $\int_0^2 (1 + 2x)\,dx$ from the formula for the area of a trapezoid or from the formulas for areas of rectangles and triangles. **(b)** Use the Riemann-sum program to calculate midpoint Riemann sums for $\int_0^2 (1+2x)\,dx$ with N equal subintervals for various positive integers N. All such Riemann sums give the exact value of the integral. Explain..

[C] **49.** **(a)** Explain why $\int_{-2}^2 x^3\,dx$ equals zero. **(b)** Use the Riemann-sum program to calculate midpoint Riemann sums for $\int_{-2}^2 x^3\,dx$ with N equal subintervals for various positive integers N. Why do all such Riemann sums give the exact value of the integral?

50.[A] Show, by writing the sum twice, once forward and once backwards, and adding corresponding terms, that for any positive integer N,

$$1 + 2 + 3 + \cdots + N = \tfrac{1}{2}N^2 + \tfrac{1}{2}N. \qquad \textbf{(12)}$$

51.[A] **(a)** Show that for any positive integer N,

$$\sum_{j=1}^{N} [(j+1)^3 - j^3] = \begin{cases} N^3 + 3N^2 + 3N \\ \displaystyle\sum_{j=1}^{N} (3j^2 + 3j + 1). \end{cases}$$

(b) Equate the expressions on the right of the last equation and use **(11)** and **(12)** to show that

$$1^2 + 2^2 + 3^2 + \cdots + N^2 = \tfrac{1}{3}N^3 + \tfrac{1}{2}N^2 + \tfrac{1}{6}N. \qquad \textbf{(13)}$$

52.[A] Derive the summation formula,

$$1^3 + 2^3 + 3^3 + \cdots + N^3 = \tfrac{1}{4}N^4 + \tfrac{1}{2}N^3 + \tfrac{1}{4}N^2 \qquad \textbf{(14)}$$

from **(12)** by the following argument:[†] The square of width $1^3 + 2^3 + 3^3 + \cdots + N^3$ can be divided into a square of width 1 and L-shaped GNOMONS of widths $2, 3, \ldots, N$ as in Figure 23. Show that the jth gnomen has area j^3 (Figure 24), so that the area of the large square is the sum in **(14)**.

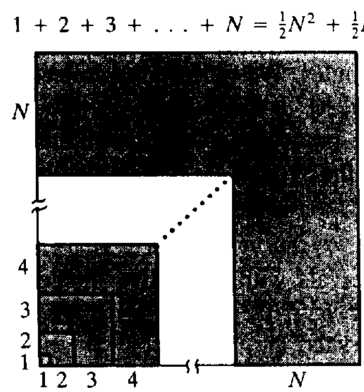


FIGURE 23

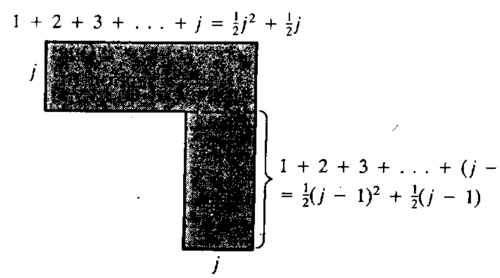


FIGURE 24

In Exercises 53 through 56 use Definition 1 and formulas **(12)** through **(14)** to find the exact values of the integrals.

53.[O] $\displaystyle\int_0^1 x\,dx$

54.[A] $\displaystyle\int_0^b x\,dx$ for arbitrary $b > 0$

55. $\displaystyle\int_0^1 x^2\,dx$

56. $\displaystyle\int_0^b x^3\,dx$ for arbitrary $b > 0$

57. Use the formula $\displaystyle\sum_{j=1}^{N} j^5 = \tfrac{1}{6}N^6 + \tfrac{1}{2}N^5 + \tfrac{5}{12}N^4 - \tfrac{1}{12}N^2$ to evaluate $\displaystyle\int_0^1 x^6\,dx$.

(End of Section 6.2)

[†]This derivation was known to Arab mathematicians in the eleventh century. (See *Extrait du Fakhrî par Alkarkhî* by F. Woepcke, Paris: l'Imprimerie Imperériale, 1853, p. 61.)