Section 6.3

(3/20/08)

The Fundamental Theorem, Part I

OVERVIEW: *The Fundamental Theorem of Calculus shows that differentiation and integration are, in a sense, inverse operations. It is presented in two parts. We previewed Part I in Section 6.1 and prove it in this section. It deals with integrals of derivatives. Part II, which will be covered in Section 6.4, involves derivatives of integrals*

Topics:

- ***Another look at Section 6.1***
- ***The Fundamental Theorem of Calculus, Part I***

Another look at Section 6.1

In the first section of this chapter we used the following result in applications involving continuous functions whose derivatives are step functions.

Theorem 1 *Suppose that a function $y = F(x)$ is continuous on a finite closed interval $[a, b]$ and that its derivative $r = F'(x)$ is a step function on $[a, b]$. Then the region between the graph $r = F'(x)$ and the x-axis for $a \le x \le b$ consists of a finite number of rectangles, and the change in the function's value from $x = a$ to $x = b$ is given by*

$$F(b) - F(a) = \left[\begin{array}{c}\textit{The area of}\\ \textit{all rectangles}\\ \textit{above the } x\textit{-axis}\end{array}\right] - \left[\begin{array}{c}\textit{The area of}\\ \textit{all rectangles}\\ \textit{below the } x\textit{-axis}\end{array}\right]. \qquad \textbf{(1)}$$

For the continuous function $y = F(x)$ whose derivative is the step function in Figure 1, for instance, this theorem states that $F(b) - F(a)$ is equal to the area of the two rectangles above the x-axis minus the area of the three rectangles below the x-axis.

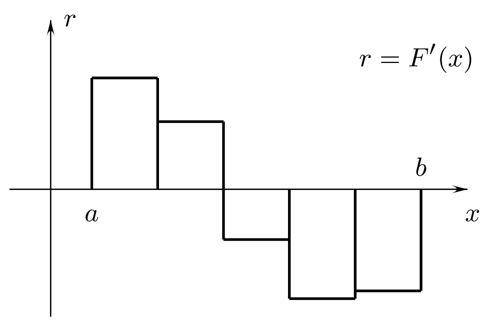

FIGURE 1

The Fundamental Theorem of Calculus, Part I

We know from Theorem 2 of Section 6.2 that the difference of areas described in Theorem 1 is equal to the integral of $F'(x)$ from a to b. Consequently, equation **(1)** can be rewritten in the form,

$$F(b) - F(a) = \int_a^b F'(x)\,dx. \qquad \textbf{(2)}$$

Equation **(2)** for functions with step-function derivatives is a special case of the following general result.

Theorem 2 (The Fundamental Theorem of Calculus, Part I)
Suppose that $y = F(x)$ is continuous and its derivative $r = F'(x)$ is piecewise continuous on an interval containing a and b. Then

$$\int_a^b F'(x)\,dx = F(b) - F(a) \tag{3a}$$

or, in Leibniz notation

$$\int_a^b \frac{dF}{dx}\,dx = F(b) - F(a). \tag{3b}$$

Proof: We can establish this theorem for general functions that satisfy its hypotheses by using the Mean Value Theorem from Section 4.1.

Recall that the Mean Value Theorem states that if $y = F(x)$ is continuous on an interval $[a, b]$ and its derivative exists for all x with $a < x < b$, then there is at least one point c with $a < c < b$ such that the average rate of change $\dfrac{F(b) - F(a)}{b - a}$ of F for $a \le x \le b$ equals its (instantaneous) rate of change $F'(c)$ at that point:

$$\frac{F(b) - F(a)}{b - a} = F'(c). \tag{4}$$

The geometric interpretation of the Mean Value Theorem is illustrated in Figure 2. Since the average rate of change of F is the slope of the secant line through the points at $x = a$ and $x = b$ on the graph and $F'(c)$ is the slope of the tangent line at $x = c$, the theorem states that there is at least one point c where the tangent line is parallel to the secant line.

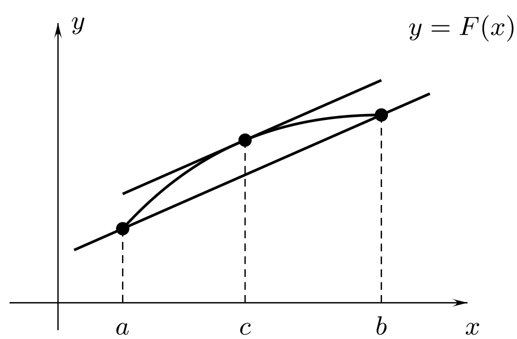

FIGURE 2

To put **(4)** in the form we need, we multiply both sides by the length $b - a$ of the x-interval. This yields

$$F(b) - F(a) = F'(c)(b - a). \tag{5}$$

To establish Theorem 2, we suppose first that a is less than b and that $F'(x)$ exists for all x with $a < x < b$. We consider an arbitrary partition,

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

of $[a, b]$. For each j we write, as usual, Δx_j for the change $x_j - x_{j-1}$ in x across the jth subinterval. The Mean Value Theorem **(5)** with $a = x_{j-1}, b = x_j, c = c_j$, and $x_j - x_{j-1} = \Delta x_j$ gives

$$F(x_j) - F(x_{i-1}) = F'(c_j)\Delta x_j \text{ for } j = 1, 2, \ldots, N. \qquad \textbf{(6)}$$

To find the change $F(b) - F(a)$ across $[a, b]$, we add the changes **(6)** across all of the subintervals. This gives

$$F(b) - F(a) = \sum_{j=1}^{N} [F(x_j) - F(x_{j-1})] = \sum_{j=1}^{N} F'(c_j)\Delta x_j. \qquad \textbf{(7)}$$

The sum on the right of **(7)** is a Riemann sum for the integral $\int_a^b F'(x)\,dx$ corresponding to the arbitrary partition of $[a, b]$. Thus, we have shown that for each partition there is at least one Riemann sum that equals $F(b) - F(a)$. Since F' is piecewise continuous on $[a, b]$, the integral is defined and equals the limit of all Riemann sums as the number of subintervals in the partitions tends to ∞ and their widths to zero. Consequently, the integral equals $F(b) - F(a)$, as is stated in **(3)**.

If F is continuous on $[a, b]$ and $F'(x)$ is defined on all of (a, b) except at one point d with $a < d < b$, we can apply the argument above to the intervals $[a, d]$ and $[d, b]$ to conclude that

$$F(b) - F(a) = [F(d) - F(a)] + [F(b) - F(d)] = \int_a^d F'(x)\,dx + \int_d^b F'(x)\,dx = \int_a^b F'(x)\,dx.$$

This argument can be repeated to establish **(3)** in any case where $F'(x)$ does not exist at a finite number of points in (a, b).

Equation **(3)** for $b < a$ follows from that equation for $a < b$ since $\int_a^b F'(x)\,dx = -\int_b^a F'(x)\,dx$ and $F(b) - F(a) = -[F(a) - F(b)]$. **QED**

Example 1 **(a)** Show that the hypotheses of Theorem 2 are satisfied for

$$F(x) = \begin{cases} 3x+2 & \text{for} \quad 0 \le x \le 1 \\ 6-x & \text{for} \quad 1 < x \le 3 \end{cases}$$

on the interval $[0,3]$. **(b)** Show, without using the theorem, that the conclusion of the theorem holds in this case by using areas to evaluate $\int_0^3 F'(x)\,dx$.

SOLUTION **(a)** The graph of $y = F(x)$ is shown in Figure 3. The function is continuous on $[0,3]$ because the polynomials $y = 3x+2$ and $y = 6-x$ are continuous for all x and both have the value 5 at $x = 1$. The derivative $r = F'(x)$, whose graph is shown in Figure 4, equals $\frac{d}{dx}(3x+2) = 3$ for $0 < x < 1$ and equals $\frac{d}{dx}(6-x) = -1$ for $1 < x < 3$. The hypotheses of Theorem 2 are satisfied because F is continuous and F' is piecewise continuous on $[0,3]$.

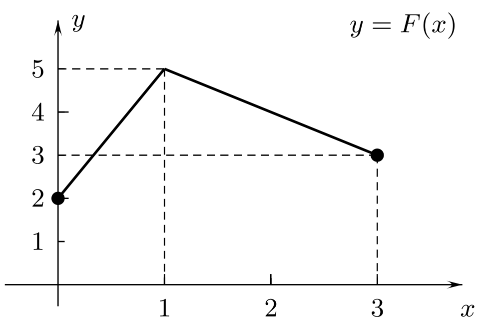

FIGURE 3

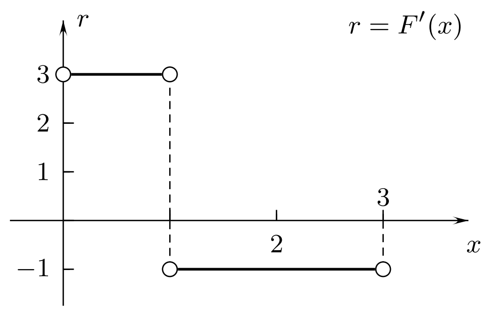

FIGURE 4

(b) The integral of $F'(x)$ from 0 to 3 equals the area of the rectangle above the x-axis in Figure 4, minus the area of the rectangle below the x-axis: $\int_0^3 F'(x)\,dx$ $= (3)(1) - (1)(2) = 1$. On the other hand, Figure 3 shows that $F(3) - F(0) = 3 - 2 = 1$. Therefore, $\int_0^3 F'(x)\,dx$ and $F(3) - F(0)$ are equal, and the conclusion of Theorem 2 is valid. □

Example 2 What is the value of $\int_0^2 \frac{d}{dx}(x^4+5)\,dx$?

SOLUTION By formula **(3b)** with $F(x) = x^4 + 5$,

$$\int_0^2 \frac{d}{dx}(x^4+5)\,dx = \Big[x^4+5\Big]_{x=2} - \Big[x^4+5\Big]_{x=0}$$
$$= [2^4+5] - [0^4+5] = 2^4 = 16. \ \square$$

Example 3 Figure 5 shows the graph of the continuous derivative $r = G'(x)$ of a continuous function $y = G(x)$. Region A in the drawing has area 89 and region B has area 62. What is $G(6) - G(1)$?

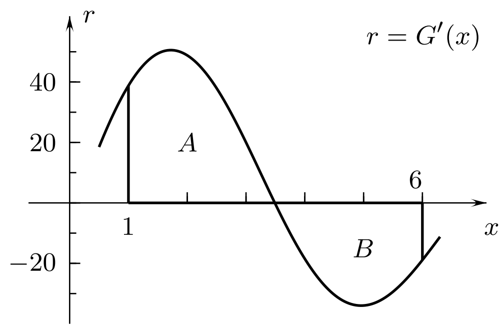

FIGURE 5

SOLUTION The Fundamental Theorem **(3a)** with G in place of F gives

$$G(6) - G(1) = \int_1^6 G'(x)\,dx.$$

By Theorem 2 of Section 6.2, the integral $\int_1^6 G'(x)\,dx$ is equal to the area of region A in Figure 7 between the graph of G' and the x-axis for $1 \le x \le 6$ where $G'(x)$ is ≥ 0, minus the area of region B between the graph and the x-axis for $1 \le x \le 6$, where $G'(x)$ is ≤ 0. Therefore,

$$G(6) - G(1) = [\text{Area of } A] - [\text{Area of } B] = 89 - 62 = 27. \square$$

It is frequently convenient to use the Fundamental Theorem in the form,

$$F(b) = F(a) + \int_a^b F'(x)\,dx. \qquad \textbf{(8)}$$

This equation is obtained by adding $F(b)$ to both sides of equation **(3a)**. Remember this formula in words: the value of F at b is equal to its value at a plus the integral of its derivative from a to b.

Example 4 A tank contains 100 gallons of water at $t = 0$ (minutes) and water is added at the rate of $20t$ gallons per minute at time t for $0 \le t \le 4$. How much water is in the tank at $t = 4$?

SOLUTION We let $V(t)$ be the volume of water at time t, measured in gallons. Then $V(0) = 100$ and $V'(t) = 20t$. We assume that V is continuous on $[0, 4]$. The Fundamental Theorem **(8)** with F replaced by V and x replaced by t reads

$$V(4) = V(0) + \int_0^4 V'(t)\,dt = 100 + \int_0^4 20t\,dt. \tag{9}$$

To find the value of the integral, we draw the graph of $V'(t) = 20t$ as in Figure 6. Because $20t$ is ≥ 0 for $0 \le t \le 4$, the integral is equal to the area of the triangle between the graph and the t-axis for $0 \le t \le 4$. Since the triangle has width 4 and height 80, its area is $\frac{1}{2}(4)(80) = 160$. With this value, equation **(9)** gives $V(4) = 100 + 160 = 260$. The tank has 260 gallons of water in it at $t = 4$. □

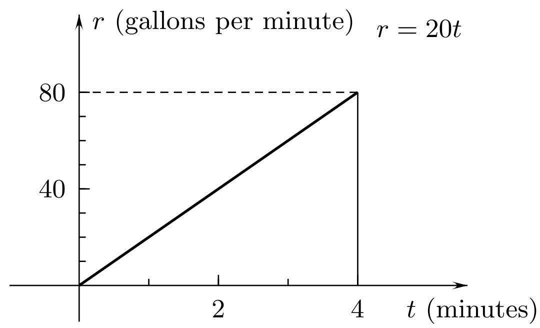

FIGURE 6

Interactive Examples 6.3

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.†

1. Evaluate the integral $\displaystyle\int_{-5}^{5} \frac{d}{dx}\left(\frac{x}{x^2+1}\right) dx$.

2. What is $W(300)$ if W is continuous and W' is piecewise continuous on $[0, 300]$, $W(0) = 7$, and $\displaystyle\int_0^{300} W'(x)\,dx = 10$?

† In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

3. Figure 7 shows the graph of the rate of CO_2 emissions by China, measured in gigatons per year, from 1990 to 2005.[1] (A gigaton is 10^5 tons.) Use the graph to find the approximate total CO_2 emissions by China from the beginning of 1990 to the beginning of 2005.

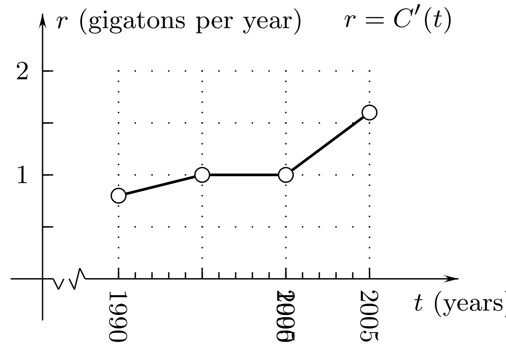

FIGURE 7

4. The graph in Figure 8 gives the rate of change with respect to time of the average price of gasoline in San Diego in January and February, 2008.[2] The price P is measured in cents and the time t is measured in days with $t = 0$ at the beginning of January 1 and $t = 60$ at the beginning of February 29. Region A in the drawing has area 8.6, region B has area 39.5, and region C has area 37.9. The price was 3.29 dollars per gallon at the beginning of January 1. What was it at the beginning of February 29?

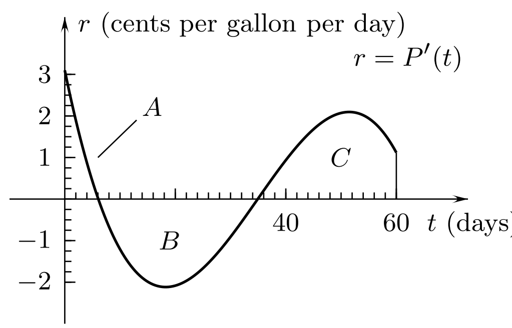

FIGURE 8

[1] Data adapted from "Climate Change—the Chinese Challenge", *Science Magazine*, February 8, 2008, p. 730.
[2] Data adapted from *San Diego Union Tribune*, February 27, 2008.

Exercises 6.3

[A]Answer provided. [O]Outline of solution provided. [C]Graphing calculator or computer required.

CONCEPTS:

1 Based on Theorem 2, what is $G(10)$ if G is continuous on $[-10, 10]$, $G(-10) = 15$ and $G'(x) = 0$ for $-10 < x < 10$?

2. What are the units in the calculation, $\frac{1}{2}(4)(80) = 160$ in the solution of Example 4?

3. Figures 9 and 10 show the graphs of a function $y = f(x)$ and its derivative $r = f'(x)$. If Theorem 2 applied in this case, then $f(3) - f(0)$ would equal $\int_0^3 f'(x)\, dx$. Show that the conclusion is not correct and explain why this is possible.

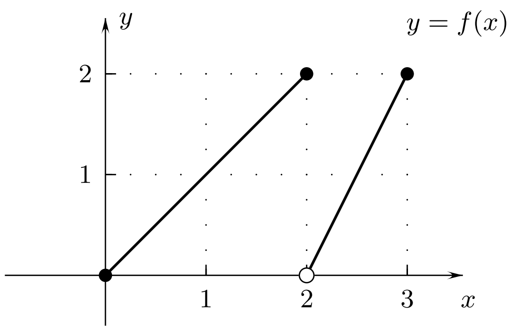

FIGURE 9

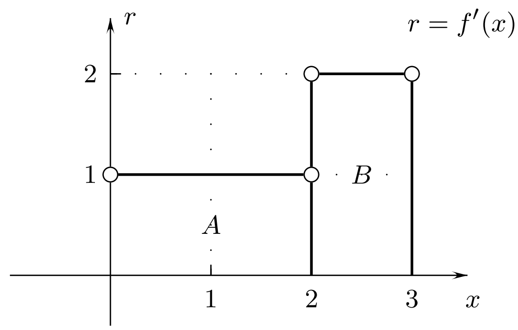

FIGURE 10

BASICS:

Evaluate the integrals in Exercises 4 through 7.

4.[O] $\displaystyle\int_0^{\pi/2} \frac{d}{dx}(\sin x)\ dx$

5.[A] $\displaystyle\int_0^{5} \frac{d}{dx}\left(\sqrt{x^2 + 144}\right)\ dx$

6. $\displaystyle\int_0^{1} \frac{d}{dx}[(x^3 + 1)^4]\ dx.$

7. $\displaystyle\int_0^{\pi/4} \frac{d}{dx}(\sin x)\ dx$

8.[O] What is $f(2)$ if f is continuous and f' is piecewise continuous on $[2, 6]$, $\int_2^6 f'(x)\ dx = 5$, and $f(6) = 8$?

9.[A] What is $Z(4) - Z(3)$ if $y = Z(t)$ is continuous and $Z'(t) = 4t^3$ for all t?

10. What is $f(10)$ if f is continuous and f' is piecewise continuous on $[0, 10], f(0) = 13$, and $\int_0^{10} f'(x)\ dx = 7$?

11. What is $G(4)$ if $G(x)$ is continuous and G' is piecwise continuous on $[-4, 4], G(-4) = 3$, and $\displaystyle\int_{-4}^{4} \frac{dG}{dx}\ dx = 0$?

12.$^{\text{O}}$ An object moving on an s-axis with coordinates given in feet is at $s = 10$ at time $t = 1$. The graph of its velocity $v(t) = s'(t)$ in the positive s-direction is shown in Figure 11. The area of region A in the drawing is 16.5 and the area of region B is 6.75. Where is the object at $t = 4$?

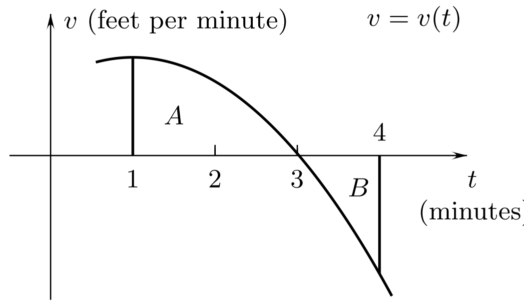

FIGURE 11

13. The temperature in a room $T = T(t)$ is 60°F at time $t = 1$ (hours). The graph of its rate of change with respect to t is shown in Figure 12. The area of region A in the drawing is 10.9 and the area of region B is 15.4 . What is the temperature in the room at $t = 4$?

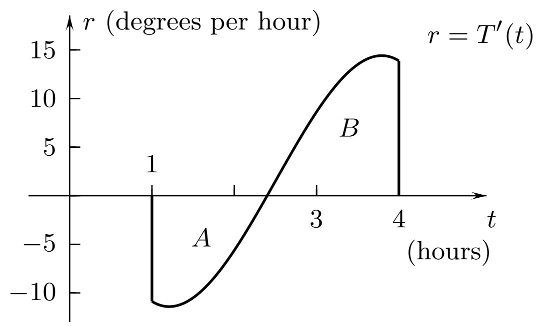

FIGURE 12

14. Figure 13 shows the graph of the rate $r = F'(t)$ of real estate foreclosures in San Diego County as a function of the time.[3] Region A in the drawing has area 5,892; region B has area 5,164; and region C has area 3,405. Based on this data how many foreclosures where there in San Diego County between the beginning of 1992 and the beginning of 2007?

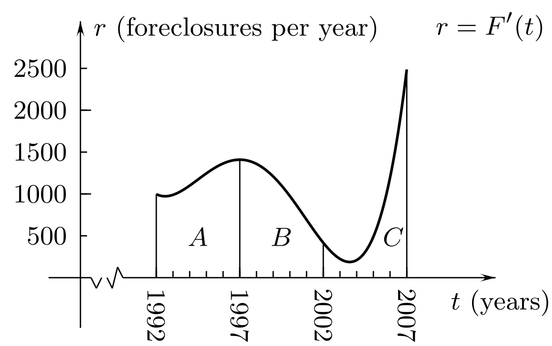

FIGURE 13

[3] Data adapted from *San Diego Union Tribune*, January 23, 2008. Source:DataQuick Information Systems.

15. The rate of change (US cents per month) of the cost of a european euro during 2007 is shown in Figure 14, where $t = 0$ at the beginning of the year.[4] One euro was worth \$1.29 at the beginning of the year. What was it worth at the end of the year?

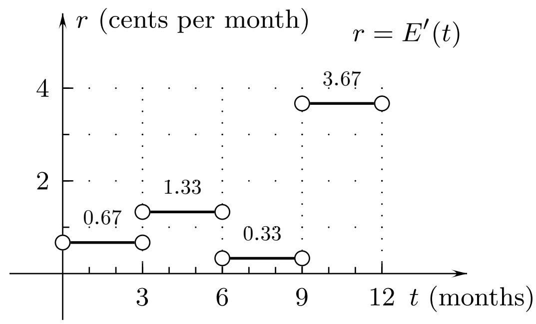

FIGURE 14

EXPLORATION:

16.[A] The derivative $y = F'(x)$ of a function $y = F(x)$ is such that the area of region A in Figure 15 is 179 and the area of region B is 396. What are the maximum and minimum values of $F(x)$ for $0 \le x \le 4$ if $F(0) = 500$?

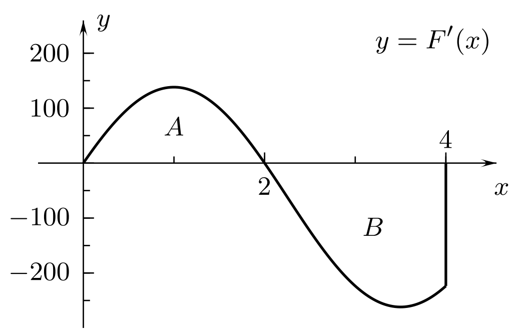

FIGURE 15

17. Figure 15 shows the graph of the rate of change of the tide level in San Diego on February 20, 2008, with $t = 0$ at midnight the previous night.[5] The tide level at midnight was 3 feet. Region A in the drawing has area 8, region B has area 6.5, region C has area 6.2, region D has area 6.5, and region D has area 8. What were the low and high tides that day?

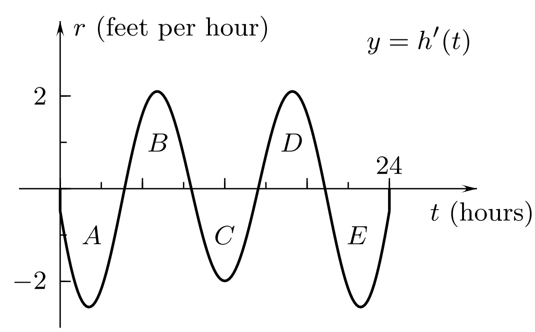

FIGURE 16

18.[A] A toy rocket is shot straight up from the ground at $t = 0$ (seconds) with an upward velocity of 64 feet per second. If there were no air resistance, the rocket's upward velocity would be $v(t) = 64 - 32t$ feet per second at time t until it hits the ground. How high is the rock at $t = 3$, assuming that there is no air resistance?

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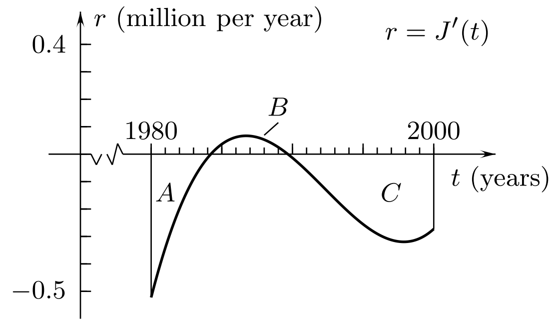
[4] Data adapted from "Foreign Exchange Rates", Federal Reserve, 2008.

[5] Data adapted from *San Diego 2008 Tide Calendar*, Tidelines, 2008.

19. Figure 17 shows the graph of the rate of change of the number of manufacturing employees in the United States from 1980 to 2000.[6] Region A in the drawing has area 0.89, region B has area 0.24, and region C has area 2.01. There were 16.65 million manufacturing employees in the U. S. at the beginning of 2000. How many were there are the beginning of 1980?

FIGURE 17



(End of Section 6.3)

[6] Data adapted from "Annual Survey of Manufactures", U.S. Census Bureau, 2008.