Section 6.4

(3/20/08)

The Fundamental Theorem, Part II

OVERVIEW: *We discussed Part I of the Fundamental Theorem of Calculus in the last section. We establish Part II of the theorem here. We also show how Part II can be used to prove Part I and how it can be combined with the Chain Rule to find derivatives of integrals with functions as limits of integration. Then we discuss a definition of the natural logarithm as an integral.*

Topics:

- ***The Fundamental Theorem, Part II***
- ***Another proof of Part I of the Fundamental Theorem***
- ***Derivatives of integrals with functions as limits of integration***
- ***Defining the natural logarithm as an integral***

The Fundamental Theorem, Part II

Part I of the Fundamental Theorem of Calculus that we discussed in Section 6.3 states that if F is continuous and its derivative F' is piecewise continuous on an interval containing a and b, then the integral of F' from a to b equals the change in F across the interval:

$$\int_a^b F'(x)\,dx = F(b) - F(a). \tag{1}$$

Part II of the Fundamental Theorem deals with derivatives of definite integrals with respect to variable upper limits of integration:

Theorem 1 (The Fundamental Theorem, Part II) *If f is continuous on an open interval I containing the point a, then the function $\int_a^x f(t)\,dt$ is differentiable on I and for all x in I,*

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

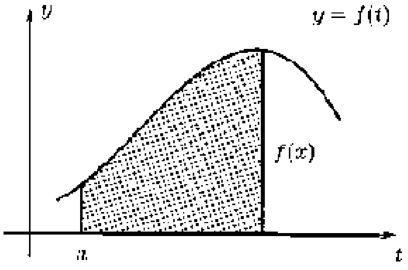
Proof: We define

$$F(x) = \int_a^x f(t)\,dt$$

for x in the interval I. We will prove the theorem by showing that for all x in I,

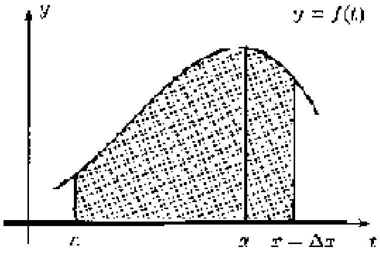
$$F'(x) = f(x). \tag{3}$$

We suppose first that $f(t)$ is nonnegative on I and that x is greater than a, as in Figure 1. Then $F(x)$ is the area of the region between the graph of $y = f(t)$ and the t-axis for $a \le t \le x$ and **(2)** states that the rate of change $F'(x)$ of the area with respect to x equals the height $f(x)$ of the region at its right side.



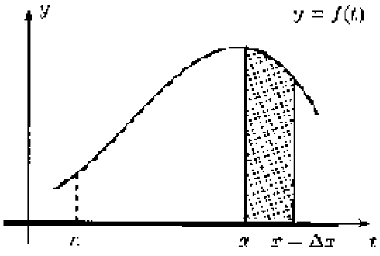
Region of area
$F(x)$

FIGURE 1



Region of area
$F(x + \Delta x)$

FIGURE 2



Region of area
$F(x + \Delta x) - F(x)$

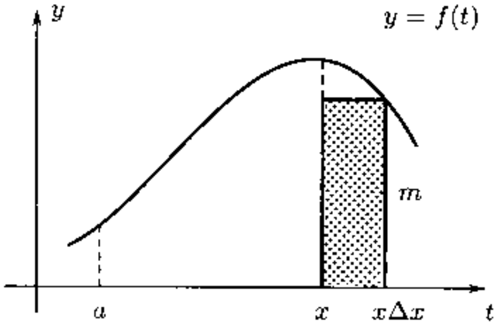
FIGURE 3

For small positive Δx, $F(x + \Delta x) = \int_a^{x+\Delta x} f(t)\,dt$ is the area of the region between the graph and the t-axis for $a \leq t \leq x + \Delta x$ in Figure 2, so that the difference

$$F(x + \Delta x) - F(x) = \int_x^{x+\Delta x} f(t)\,dt$$

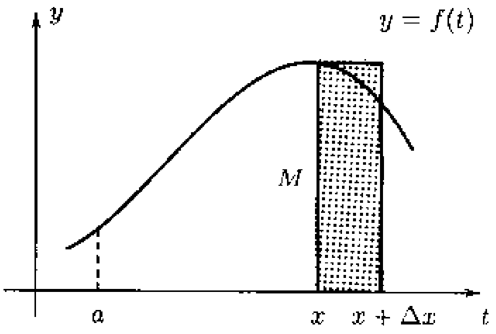
is the area of the region between the graph and the t-axis for $x \leq t \leq x + \Delta x$ in Figure 3.

Because f is continuous on the finite closed interval $x \leq t \leq x + \Delta x$, it has—by the Extreme Value Theorem of Section 3.2—a minimum value m (Figure 4) and a maximum value M (Figure 5) in the interval.



Rectangle of area $m\Delta X$

FIGURE 4



Rectangle of area $M\Delta X$

FIGURE 5

The rectangle in Figure 4 has area $m\Delta x$ and is contained in the region of Figure 3, while the rectangle in Figure 5 has area $M\Delta x$ and contains the region of Figure 3. Therefore,

$$m\Delta x \leq F(x + \Delta x) - F(x) \leq M\Delta x.$$

Dividing all the expressions in these inequalities by the positive number Δx yields

$$m \leq \frac{F(x + \Delta x) - F(x)}{\Delta x} \leq M. \tag{4}$$

Similar reasoning would give **(4)** for small negative Δx. The numbers M and m are values of $y = f(t)$ in the interval between x and $x + \Delta x$. Because $y = f(t)$ is continuous at x, m and M both tend to $f(x)$ as Δx tends to 0, and **(4)** shows that

$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = f(x).$$

This gives **(3)** to prove the theorem in this case. Similar reasoning yields the theorem for $x < a$ and for f that has negative values. **QED**

Example 1 Find the derivative $\dfrac{d}{dx}\displaystyle\int_1^x \sqrt{t^4+7}\,dt$.

SOLUTION Because $y = \sqrt{t^4+7}$ is continuous for all t, formula **(2)** shows that

$$\frac{d}{dx}\int_1^x \sqrt{t^4+7}\,dt = \left[\sqrt{t^4+7}\right]_{t=x} = \sqrt{x^4+7}. \square$$

Another proof of Part 1 of the Fundamental Theorem

We can now use Part II of the Fundamental Theorem above to give another proof of Part I, which was established in Section 6.3 by using the Mean Value Theorem.

Theorem 2 (The Fundamental Theorem of Calculus, Part I) *If F is continuous and its derivative F' is piecewise continuous on an interval I containing a and b, then*

$$\int_a^b F'(x)\,dx = F(b) - F(a). \tag{5}$$

Proof: We consider the case where the interval I is open and F' is continuous on it. We define

$$G(x) = \int_a^x F'(t)\,dt \tag{6}$$

for x in I. By Part II of the Fundamental Theorem above, $G'(x) = F'(x)$ for all x in I. With the Mean Value Theorem this implies that

$$G(x) = F(x) + C \text{ for all } x \text{ on } I \tag{7}$$

with some constant C. By definition **(6)**, $G(a) = \displaystyle\int_a^a F'(x)\,dx = 0$, so that $F(a) + C = 0$ and $C = -F(a)$. Setting $x = b$ and $C = -F(a)$ in **(7)** gives $G(b) = F(b) - F(a)$, which in turn yields **(5)**. **QED**

Derivatives of integrals with functions as limits of integration

We will now see how Theorem 1 can be used to find the x-derivatives of other integrals whose upper and/or lower limits of integration are functions of x.

To apply Theorem 1 when the upper limit of integration is constant and the lower limit is x, we first interchange the limits of integration and, to preserve the value of the integral, multiply it by -1.

Example 2 What is the x-derivative of $\int_x^5 e^{\sqrt{t}}\,dx$ at $x = 4$?

SOLUTION Switch the limits of integration, insert a minus sign, and then use Theorem 1:

$$\frac{d}{dx}\int_x^5 e^{\sqrt{t}}\,dt = \frac{d}{dx}\left[-\int_5^x e^{\sqrt{t}}\,dt\right] = -e^{\sqrt{x}}.$$

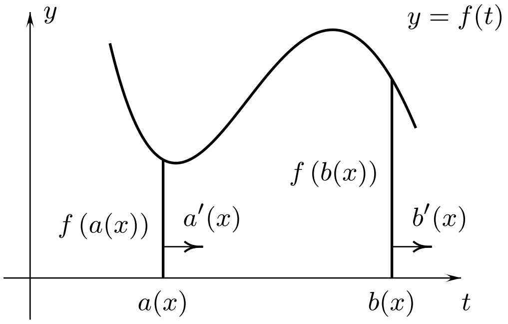
The derivative at $x = 4$ is $-e^{\sqrt{4}} = -e^2$. □

If the limits of integration are functions other than $y = x$, we use the following generalization of Theorem 1.

Theorem 3 *If f is continuous on an open interval I containing the values of differentiable functions $y = a(x)$ and $y = b(x)$, then*

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f\left(b(x)\right)\,b'(x) - f\left(a(x)\right)\,a'(x). \tag{8}$$

Theorem 3 has the geometric interpretation in Figure 6 in the case where f is a positive function and $a(x)$ is less than $b(x)$. The integral $\int_{a(x)}^{b(x)} f(t)\,dt$ in **(8)** equals the area of the region between the graph of $y = f(t)$ and the t-axis for $a(x) \le t \le b(x)$. Equation **(8)** states that the rate of change of this area equals the height $f\left(b(x)\right)$ of the right side multiplied by the rate $b'(x)$ at which it is moving to the right, minus the height $f\left(a(x)\right)$ of the left side multiplied by the rate $a'(x)$ at which it is moving to the right.



[Rate of change of the area]

$= f\left(b(x)\right) b'(x) - f\left(a(x)\right) a'(x)$

FIGURE 6

Proof of Theorem 3: Let t_0 be any point in the interval I and define

$$F(u) = \int_{t_0}^{u} f(t)\,dt.$$

Then $F'(u) = f(u)$ for u in I by Theorem 1. The integral from $t = a(x)$ to $t = b(x)$ equals the integral from $a(x)$ to t_0 plus the integral from t_0 to $b(x)$. Therefore, by the Chain Rule,

$$\begin{aligned}
\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt &= \frac{d}{dx}\left(\int_{a(x)}^{t_0} f(t)\,dt + \int_{t_0}^{b(x)} f(t)\,dt\right)\\
&= \frac{d}{dx}\left(-\int_{t_0}^{a(x)} f(t)\,dt + \int_{t_0}^{b(x)} f(t)\,dt\right)\\
&= \frac{d}{dx}[-F(a(x)) + F(b(x))] = -F'(a(x))\,a'(x) + F'(b(x))\,b'(x)\\
&= f(b(x))\,b'(x) - f(a(x))\,a'(x).\ \textbf{QED}
\end{aligned}$$

Example 3 What is the x-derivative of $\displaystyle\int_0^{x^2} \cos t\,dt$?

SOLUTION By **(8)** with $f(t) = \cos t, a(x) = 0$, and $b(x) = x^2$,

$$\frac{d}{dx}\int_0^{x^2} \cos t\,dt = \Big[\cos t\Big]_{t=x^2}\frac{d}{dx}(x^2) = 2x\cos(x^2).\ \square$$

Example 4 Find $F'(2)$ for $F(x) = \displaystyle\int_{4x}^{x^4} e^t\,dt$.

SOLUTION We use **(8)** with $f(t) = e^t, a(x) = 4x$, and $b(x) = x^4$:

$$\begin{aligned}
F'(x) &= \frac{d}{dx}\int_{4x}^{x^4} e^t\,dt\\
&= \Big[e^t\Big]_{t=x^4}\frac{d}{dx}(x^4) - \Big[e^t\Big]_{t=4x}\frac{d}{dx}(4x)\\
&= 4x^3e^{x^4} - 4e^{4x}.
\end{aligned}$$

Therefore, $F'(2) = 4(2^3)e^{2^4} - 4e^{4(2)} = 32e^{16} - 4e^8$. $\square$

Defining the natural logarithm as an integral

Our discussion of derivatives of exponential functions and logarithms in Sections 3.2 and 3.3 used one fact that we left unproved because its direct derivation is relatively complicated: we did not show that the limit $e = \lim_{t\to\infty}(1+1/t)^t$ defining the number e exists. This is established in the next theorem.

Theorem 4 *The limit*

$$e = \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t \tag{9}$$

exists and the logarithm to the base e is given by the integral,

$$\ln x = \int_1^x \frac{1}{t}\,dt \ \text{ for } x > 0. \tag{10}$$

Proof: We define

$$L(x) = \int_1^x \frac{1}{t}\,dt \ \text{ for } x > 0. \tag{11}$$

By Part II of the Fundamental Theorem, this function is differentiable and hence continuous on $(0,\infty)$ and its derivative is

$$L'(x) = \frac{1}{x} \ \text{ for all } \ x > 0. \tag{12}$$

Moreover, because the limits of integration in **(11)** are equal for $x = 1$, we have

$$L(1) = 0. \tag{13}$$

Consider an arbitrary rational constant n. The formula $\frac{d}{dx}(x^n) = nx^{n-1}$ from Section 2.4 with **(12)** and the Chain Rule gives

$$\frac{d}{dx}L(x^n) = L'(x^n)\frac{d}{dx}(x^n) = \frac{nx^{n-1}}{x^n} = \frac{n}{x}.$$

This equation with **(12)** shows that $y = L(x^n)$ and $y = nL(x)$ have the same derivative for $x > 0$. Therefore, $L(x^n) = nL(x) + C$ for $x > 0$ with some constant C. Setting $x = 1$ gives $C = 0$, so that

$$L(x^n) = nL(x) \text{ for } x > 0. \tag{14}$$

We need to show that **(14)** also holds with the rational n replaced by an irrational number t. Recall that if t is irrational, then x^t is defined by $x^t = \lim_{n\to\infty} x^{r_n}$ where r_n are rational numbers that tend to t as $n \to \infty$. Because L is continuous, formula **(14)** for each r_n gives $L(x^t) = \lim_{n\to\infty} L(x^{r_n}) = \lim_{n\to\infty} r_n L(x) = tL(x)$. This shows that for any number t, rational or irrational,

$$L(x^t) = tL(x) \text{ for } x > 0. \tag{15}$$

Because L has a positive derivative, it is increasing on $(0,\infty)$ and we can define e to be the unique number such that $L(e) = 1$. With this definition, formula **(15)** gives $L(e^x) = xL(e) = x$ for all positive x, so that $L(x) = \log_e x = \ln x$ and **(10)** holds.

Now we can use the logarithm to establish **(9)**. We consider an arbitrary positive number t and set $\Delta t = 1/t$. Properties **(13)** and **(15)** of $L(x) = \ln x$ show that

$$\ln\left((1+1/t)^t\right) = t\ln(1+1/t) = \frac{\ln(1+1/t) - \ln(1)}{1/t} = \frac{\ln(1+\Delta t) - \ln(1)}{\Delta t}.$$

Since $\Delta t \to 0$ when $t \to \infty$, the definition of the derivative gives

$$\lim_{t\to\infty} \ln\left((1+1/t)^t\right) = \lim_{\Delta t\to 0^+} \frac{L(1+\Delta t)-L(1)}{\Delta t} = \left[\frac{d}{dx}(\ln x)\right]_{x=1} = \left[\frac{1}{x}\right]_{x=1} = 1.$$

Finally, because $y = e^x$ is the inverse of $y = \ln x$ and is continuous,

$$\lim_{t\to\infty} (1+1/t)^t = \lim_{t\to\infty} e^{\ln[(1+1/t)^t]} = e^1 = e$$

as stated in **(9)**. **QED**

Interactive Examples 6.4

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.[†]

Find the derivatives in Examples 1 through 4.

1. $\dfrac{d}{dx}\displaystyle\int_{-1}^{x} t\sin t\,dt$

2. $\dfrac{d}{dx}\displaystyle\int_{x}^{2} e^{t^2}\,dt$

3. $\dfrac{d}{dx}\displaystyle\int_{1}^{x^2} (t^3+5)^{10}\,dt$

4. $\dfrac{d}{dx}\displaystyle\int_{2x}^{3x} (10+t^2)^{1/3}\,dt$

Exercises 6.4

$^{\mathsf{A}}$Answer provided. $^{\mathsf{O}}$Outline of solution provided. $^{\mathsf{C}}$Graphing calculator or computer required.

CONCEPTS:

1. Figure 7 shows the graph of the step function

$$f(t) = \begin{cases} 10 & \text{for} \quad 0 \le t < 3 \\ 20 & \text{for} \quad 3 \le t \le 5. \end{cases}$$

(a) Use the formula for the area of a rectangle to find formulas for $F(x) = \displaystyle\int_0^x f(t)\,dt$ for $0 \le x \le 5$. **(b)** Use the formulas from part (a) to give formulas for $y = F'(x)$ and draw its graph. **(c)** How do the graphs of $y = f(t)$ and $y = F'(x)$ illustrate Theorem 1?

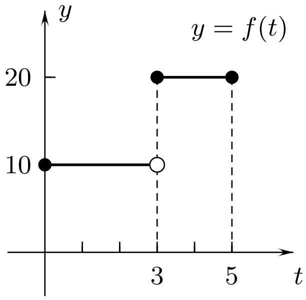


FIGURE 7

[†] In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

2. Set $P(x) = \int_0^x p(t)\,dt$, where $y = p(t)$ is the continuous function of Figure 8. Draw the graph of $y = P'(x)$

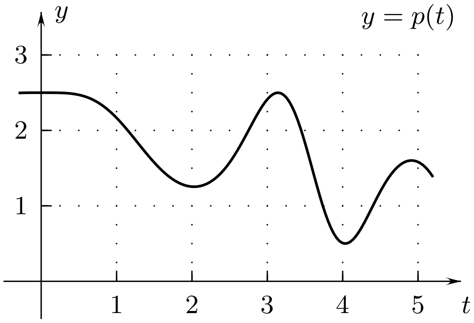


FIGURE 8

3. Give a geometric interpretation of the equation $F'(x) = f(x)$ for $F(x) = \int_a^x f(t)\,dt$ in the case of a negative continuous function f and $x > a$ as in Figure 9.

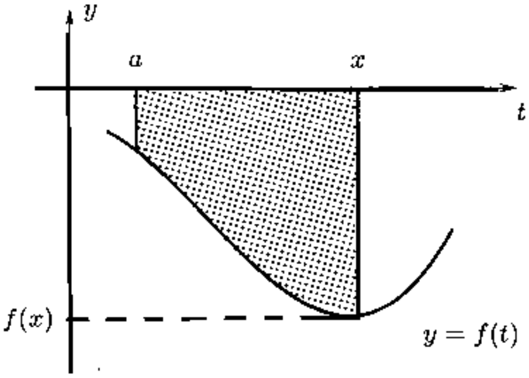


FIGURE 9

4. **(a)** Give a geometric interpretation of the equation $F'(x) = -f(x)$ for $F(x) = \int_x^a f(t)\,dt$, where $x < a$ and where $f(t)$ is positive for $x < a$, as in Figure 10. **(b)** Why is $F'(x)$ negative for $x < a$ in this case?

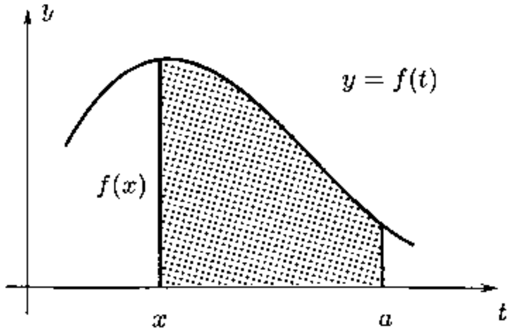


FIGURE 10

5. **(a)** Use Theorem 3 to find $F'(x)$ where $F(x) = \displaystyle\int_{a(x)}^{b(x)} 5\,dt$ with differentiable $a(x)$ and $b(x)$.

(b) Give a geometric interpretation of the result of part (a) for $a(x) < b(x)$ as in Figure 11.

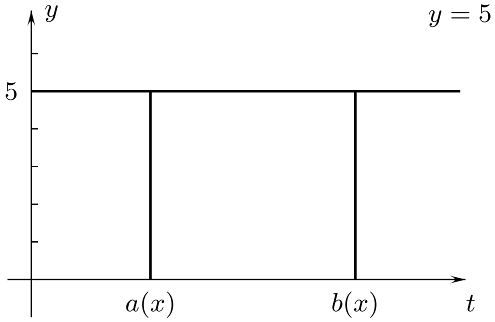


FIGURE 11

BASICS:

Find the derivatives in Exercises 6 through 23.

6.$^{\mathsf{O}}$ $\dfrac{d}{dx}\displaystyle\int_3^x \ln t\,dt$

7.$^{\mathsf{O}}$ $\dfrac{d}{dx}\displaystyle\int_x^{10} \sin(t^2)\,dt$

8.$^{\mathsf{O}}$ $\dfrac{d}{dx}\displaystyle\int_0^{5x} (t^2+3t)^{1/3}\,dt$

9.$^{\mathsf{O}}$ $\dfrac{d}{dx}\displaystyle\int_{x^2}^{x^3} \frac{1}{1+t^2}\,dt$

10.$^{\mathsf{A}}$ $\dfrac{d}{dx}\displaystyle\int_1^{x} \sqrt{t^2+10}\,dt$ at $x = 3$

11.$^{\mathsf{O}}$ $\dfrac{d}{dx}\displaystyle\int_0^{x} \sin^2 t\,dt$

12. $\dfrac{d}{dx}\displaystyle\int_0^{x} (e^{t^2}+1)\,dt$

13.$^{\mathsf{O}}$ $\dfrac{d}{dx}\displaystyle\int_x^{2} \sin(t^2+1)\,dt$

14. $\dfrac{d}{dx}\displaystyle\int_x^{0} t^{8/3}\,dt$

15.$^{\mathsf{O}}$ $\dfrac{d}{dx}\displaystyle\int_1^{2x+3} te^t\,dt$

16.$^{\mathsf{A}}$ $\dfrac{d}{dx}\displaystyle\int_1^{x^3} \frac{t}{t^4+1}\,dt$

17.$^{\mathsf{A}}$ $\dfrac{d}{dx}\displaystyle\int_{x^3}^{10} (\cos t)^{1/3}\,dt$

18. $\dfrac{d}{dx}\displaystyle\int_1^{3x} \sin^5 t\,dt$

19.$^{\mathsf{O}}$ $\dfrac{d}{dx}\displaystyle\int_{sinx}^{\cos x} e^{t^2}\,dt$

20.$^{\mathsf{A}}$ $\dfrac{d}{dx}\displaystyle\int_{x^2}^{x^3} \sin(t+1)\,dt$

21. $\dfrac{d}{dx}\displaystyle\int_{x^2+x}^{x^3+x^2} \frac{1}{t}\,dt$

22. $\dfrac{d}{dx}\displaystyle\int_{-x}^{x} \frac{1}{t^2+1}\,dt$

23. $\dfrac{d}{dx}\displaystyle\int_{\sin x}^{\cos x} (t^2+1)^2\,dt$

EXPLORATION:

24.$^{\mathsf{A}}$ For $x > 0$, the area of the region between the graph of the positive continuous function $y = f(t)$ and the t-axis for $0 \le t \le x$ is $5x^5 + x$. What is the function f?

25. What is $g(t)$ if $\dfrac{d}{dx}\displaystyle\int_0^{x^3} g(t)\,dt = 3x^2\sin(x^6+1)$?

(End of Section 6.4)