Section 6.5

(3/20/08)

Integrals of powers and indefinite integrals

OVERVIEW: *In this section we use Part I of the Fundamental Theorem of Calculus to derive a formula for integrals of powers $y = x^n$ with $n \neq -1$. Then we introduce the term "indefinite integral" for "antiderivative."*

Topics:

- ***Integrals of*** $\mathbf{y = x^n}$ ***with*** $\mathbf{n \neq -1}$
- ***Indefinite integrals***

Integrals of $\mathbf{y = x^n}$ *with* $\mathbf{n \neq -1}$

We can evaluate a definite integral of a function $y = f(x)$ if we can find a function $y = F(x)$ whose derivative is f. Such a function F is called an ANTIDERIVATIVE of f. In particular, if F is continuous and f' is piecewise continuous on an interval and $F'(x) = f(x)$ in the interior of the interval, then by Part I of the Fundamental Theorem (Theorem 2 of Section 6.3),

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a) \tag{1}$$

for any numbers a and b in the interval. We use this approach to derive the following theorem.

Theorem 1 *For any constant $n \neq -1$ and for a and b in an interval where x^n is defined,*

$$\int_a^b x^n\,dx = \left[\frac{1}{n+1}\,x^{n+1}\right]_a^b = \frac{1}{n+1}\,b^{n+1} - \frac{1}{n+1}\,a^{n+1}. \tag{2}$$

In formula **(2)** we used the symbol $\Big[F(x)\Big]_a^b$ for $F(b) - F(a)$.

Notice that **(2)** reads $\displaystyle\int_a^b 1\,dx = b - a$ for $n = 0$ and reads $\displaystyle\int_a^b x\,dx = \tfrac{1}{2}b^2 - \tfrac{1}{2}a^2$ for $n = 1$.

Theorem 1 does not apply if $n = -1$. We will see in Section 6.6 that integrals of x^{-1} are given by logarithms.

Proof of Theorem 1: Let n be any constant $\neq -1$. To apply **(1)** with $f(x) = x^n$, we need a function F whose derivative is x^n. Notice that in finding the derivative $\dfrac{d}{dx}(x^n) = nx^{n-1}$, we first multiply x^n by the exponent n and then decrease the exponent by 1. Accordingly, we should be able to find a function whose derivative is x^n by performing the opposite steps in the opposite order. We raise the exponent in x^n by 1 to $n+1$ and then divide by the new exponent to obtain $F(x) = \dfrac{1}{n+1}x^{n+1}$.

This is, in fact, an antiderivative of x^n because

$$F'(x) = \frac{d}{dx}\left(\frac{1}{n+1}x^{n+1}\right) = \frac{1}{n+1}[(n+1)x^n] = x^n. \tag{3}$$

Setting $f(x) = x^n$ and $F(x) = \dfrac{1}{n+1}\,x^{n+1}$ in **(1)** gives **(2)**. **QED**

Example 1 What is the value of the integral $\int_{-1}^{1} x^2\,dx$?

SOLUTION By **(2)** with $a = 1, b - 2$, and $n = 2$, for which $n + 1 = 3$,

$$\int_{-1}^{1} x^2\,dx = \left[\tfrac{1}{3}x^3\right]_{-1}^{1} = \tfrac{1}{3}(1^3) - \tfrac{1}{3}(-1)^3 = \tfrac{2}{3}. \ \Box$$

We can apply Theorem 1 to linear combinations $y = Ax^n + Bx^m$ of powers of x, as in the next example, by writing for $n \neq -1$ and $m \neq -1$,

$$\int_a^b (Ax^n + Bx^m)\,dx = \left[\frac{A}{n+1}x^{n+1} + \frac{B}{m+1}x^{m+1}\right]_a^b. \qquad \mathbf{(4)}$$

Example 2 Evaluate $\int_1^2 (4x^{1/3} + 6x^{-2})\,dx$.

SOLUTION By **(4)** with $n = \frac{1}{3}$, for which $n + 1 = \frac{4}{3}$ and with $m = -2$, for which $m + 1 = -1$,

$$\begin{aligned}\int_1^2 (4x^{1/3} + 6x^{-2})\,dx &= \left[\frac{4}{4/3}x^{4/3} + \frac{6}{-1}x^{-1}\right]_1^2 \\ &= \left[3x^{4/3} - 6x^{-1}\right]_1^2 \\ &= [3(2^{4/3}) - 6(2^{-1})] - [3(1^{4/3}) - 6(1^{-1})] \\ &= [3(2^{4/3}) - 3] - [3 - 6] = 3(2^{4/3}). \ \Box\end{aligned}$$

We generally apply **(4)**, as in the next example, without writing it out explicitly.

Example 3 Find the area of the region bounded by the curve $y = 3x^2 - x^3$ and the x-axis.

SOLUTION We first need to sketch the region. The factorization $3x^2 - x^3 = x^2(3 - x)$ shows that $y = 3x^2 - x^3$ is zero at $x = 0$ and at $x = 3$, is positive for $x < 0$ and for $0 < x < 3$, and is negative for $x > 3$. Consequently, its graph intersects the x-axis at $x = 0$ and at $x = 3$, is above the x-axis for $x < 0$ and for $0 < x < 3$, and is below the x-axis for $x > 3$, as is shown in Figure 1.

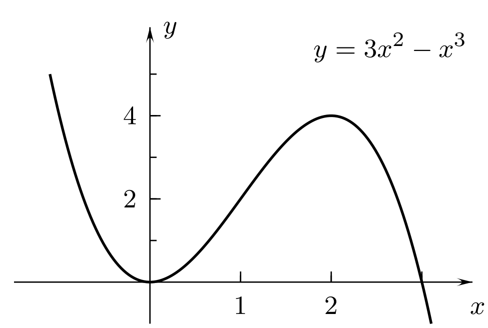

FIGURE 1

The top of the region is formed by the curve $y = 3x^2 - x^3$ and its bottom by the x-axis. It extends for $0 \le x \le 3$, so that by Definition 4 of Section 6.2, its area is

$$\int_0^3 (3x^2 - x^3)\,dx.$$

We apply the integration formula **(2)** with $n = 2$ and $n + 1 = 3$ to the first term and with $n = 3$ and $n + 1 = 4$ to the second to see that the area equals

$$\begin{aligned}\int_0^3 (3x^2 - x^3)\,dx &= \Big[3(\tfrac{1}{3}x^3) - \tfrac{1}{4}x^4\Big]_0^3 = \Big[x^3 - \tfrac{1}{4}x^4\Big]_0^3 \\ &= [3^3 - \tfrac{1}{4}(3^4)] - [0^3 - \tfrac{1}{4}(0^4)] \\ &= [27 - \tfrac{81}{4}] - [0] = \tfrac{27}{4}. \;\square\end{aligned}$$

Indefinite integrals

If we add $F(a)$ to both sides of formula **(1)** in the Fundamental Theorem, we obtain a formula,

$$F(b) = F(a) + \int_a^b F'(x)\,dx \qquad \textbf{(5)}$$

for the value of the function at one point b in terms of its value at a second point a and its derivative. This formula shows that the process of finding definite integrals is, in a sense, the inverse of the process of differentiation. This leads use to use the term INDEFINITE INTEGRALS for antiderivatives and to use the integration symbol to represent them, as in the following definition.

Definition 1 *The indefinite integral* $\int f(x)\,dx$ *is an antiderivative of* $f(x)$.

We saw in equation **(3)** that $\dfrac{1}{1+n}\,x^{n+1}$ is an antiderivative of x^n for $n \ne -1$. The function $\dfrac{1}{1+n}\,x^{n+1} + C$ is also an antiderivative of x^n for any constant C because the derivative of the constant is zero. We express this result as follows.

Theorem 2 *If* $n \ne -1$*, then in any open interval where* x^n *is defined,*

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C. \qquad \textbf{(6)}$$

Here C *can be any constant.*

The integral in **(6)** is called "indefinite" because the constant C, which is called the CONSTANT OF INTEGRATION, can be any number.

The differentiation formula $\dfrac{d}{dx}[AF(x) + BG(x)] = AF'(x) + BG'(x)$ translates into a similar formula for indefinite integrals:

$$\int [Af(x) + Bg(x)]\,dx = A\int f(x)\,dx + B\int g(x)\,dx. \qquad \textbf{(7)}$$

We usually apply **(7)** without referring to it explicitly.

Example 4 Find the antiderivative $\int \left(3\sqrt{x} + \frac{4}{x^2} - 3\right)\, dx$.

SOLUTION We rewrite the first two terms in the integrand with exponential notation by writing

$$\int \left(3\sqrt{x} + \frac{4}{x^2} - 3\right)\, dx = \int (3x^{1/2} + 4x^{-2} - 3)\, dx.$$

Then we use Theorem 3 with $n = \frac{1}{2}$ and $n + 1 = \frac{1}{2} + 1 = \frac{3}{2}$ to find an antiderivative of $x^{1/2}$ and with $n = -2$ and $n + 1 = -1$ to find an antiderivative of x^{-2}. We also use $3x$ as an antiderivative of the constant function 3, and obtain

$$\begin{aligned}\int \left(3\sqrt{x} + \frac{4}{x^2} - 3\right)\, dx &= \int (3x^{1/2} + 4x^{-2} - 3)\, dx \\ &= 3\left[\frac{1}{3/2}x^{3/2}\right] + 4\left[\frac{1}{-1}x^{-1}\right] - 3x + C \\ &= 2x^{3/2} - 4x^{-1} - 3x + C. \ \square\end{aligned}$$

We can verify the formula for the indefinite integral in Example 4 by differentiating the result. We obtain

$$\begin{aligned}\frac{d}{dx}(2x^{3/2} - 4x^{-1} - 3x) &= 2(\tfrac{3}{2})x^{1/2} - 4(-x^{-2}) - 3 \\ &= 3x^{1/2} + 4x^{-2} = 3\sqrt{x} + \frac{4}{x^2} - 3.\end{aligned}$$

This is the original integrand, so the indefinite integral is correct.

If we want to determine the value of a function at one point from its value at a second point and its derivative, we can either use an indefinite integral or use **(5)** with a definite integral. It is generally easier to use an indefinite integral if the derivative is a linear combination of powers, as is illustrated in the next example.

Example 5 Suppose that the temperature in a room is 50°F at time $t = 0$ (hours) and that the rate of change of the temperature is $r = 12t^2 - 4t^3$ degrees per hour at time t for $0 \le t \le 2$. What is the temperature at $t = 2$?

SOLUTION Since $T(t)$ is an antiderivative of its derivative, $r(t)$,

$$\begin{aligned}T(t) &= \int r(t)\, dt = \int (12t^2 - 4t^3)\, dt \\ &= 12(\tfrac{1}{3}t^3) - 4(\tfrac{1}{4}(t^4) + C = 4t^3 - t^4 + C.\end{aligned} \quad \textbf{(8)}$$

The constant of integration C in **(8)** can be determined from the given temperature of 50° at $t = 0$. Setting $t = 0$ and $T(0) = 50$ in **(5)** gives $50 = 4(0^4) - 0^3 + C$, so that $C = 50$ and by **(9)**, $T(t) = 4t^3 - t^4 + 50$. Finally, we set $t = 2$ to obtain $T(2) = 4(2^3) - 2^4 + 50 = 32 - 16 + 50 = 66$. The temperature at $t = 2$ is 66°F. $\square$

Interactive Examples 6.5

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.†

1. Find the antiderivative $\int (4 + 7x^3 - x^5)\, dx$.

2. Evaluate $\int_1^2 \left(\frac{1}{t^2} - \frac{t^2}{4}\right) dt$.

3. Find the area of the region bounded by $y = \sqrt{x} - x^2$ and the x-axis.

4. The water in a reservoir is 50 meters deep at noon, January 5 and is decreasing at the rate of $\frac{1}{10} + \frac{1}{50}t$ meters per day t days after that time. How deep is the water at noon January 15?

Exercises 6.5

[A]Answer provided. [O]Outline of solution provided. [C]Graphing calculator or computer required.

CONCEPTS:

1. Show that $\int_0^1 x^2\, dx = \int_{-1}^0 x^2\, dx$ **(a)** by geometric reasoning using symmetry and **(b)** by using Theorem 1 of this section.

2. Show that $\int_0^1 x^3\, dx = -\int_{-1}^0 x^3\, dx$ **(a)** by geometric reasoning using symmetry and **(b)** by using Theorem 1.

3. What is wrong with the equation $\int 2x\, dx = \left(\int 2\, dx\right)\left(\int x\, dx\right)$?

4. What is wrong with the calculation, $\int_0^5 x^{-2}\, dx = \left[-x^{-1}\right]_0^5 = -5^{-1} + 0^{-1}$?

BASICS:

Find formulas for the indefinite integrals and values of the definite integrals in Problems 5 through 17.

5.[O] $\int (5x^4 - 7x^6)\, dx$

6.[A] $\int (x^{1/3} + x^{1/4} + x^{1/5})\, dx$

7. $\int (9x^8 - 8x^7)\, dx$

8. $\int_0^1 (x - \sqrt{x} + \sqrt[3]{x})\, dx$

9.[A] $\int (7x^{5/7} + 5x^{7/5})\, dx$

10. $\int_1^3 (x^2 + 10x)\, dx$

11.[A] $\int_8^0 (4x^{2/3} - 3)\, dx$

12.[O] $\int_1^8 (2t + t^{-2})\, dt$

13.[A] $\int_{-4}^{-1} \left(\frac{2}{y^3} - \frac{3}{y^4}\right) dt$

14. $\int_1^3 (x^{-2} - 2x)\, dx$

15. $\int_1^9 \frac{1}{\sqrt{x}}\, dx$

16. $\int (3 + 4x - 5x^2)\, dx.$

17.[O] $\int_0^4 K(x)\, dx$ where $K(x) = \begin{cases} x^3 & \text{for} \quad x < 2 \\ x^{-2} & \text{for} \quad x > 2. \end{cases}$

18.[O] Find the function $y = W(x)$ such that $W'(x) = 3x^5$ and $W(1) = 2$.

19. What is $P(x)$ if $P'(x) = x^{1/3}$ and $P(9) = -5$?

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†In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

In Exercises 20 through 23 draw the regions and find their areas.

20.[O] The region between $y = 4 + 3x^2$ and the x-axis for $0 \le x \le 2$

21.[A] The region between $y = -6x^{-2}$ and the x-axis for $1 \le x \le 3$

22. The region between $y = -1 - \frac{1}{4}x^2$ and the x-axis for $-3 \le x \le 3$

23. The region between $y = 1 + x^2 + x^4$ and the x-axis for $0 \le x \le 1$

24.[O] The temperature in a room is 68°F at 6:00 AM and is rising $\frac{1}{60}\sqrt{t}$ degrees per minute t minutes after 6:00 AM. What is the temperature at 7:00 AM?

25.[A] A culture consist of fifteen thousand bacteria initially and is growing at the rate of $5+50t+15t^2$ thousand bacteria per hour t hours later. How many bacteria are in the culture after one hour?

26. A car is 25 miles north of a town at 1:00 PM and its velocity is $50+30t^{-2}$ miles per hour toward the north t hours after noon for $1 \le t \le 4$. Give a formula for its distance from the town t hours after noon for $1 \le t \le 4$.

27.[A] The barometric pressure in a town is 745 millimeters of mercury at time $t = 1$ (hours) and its rate of change is $2t - 6$ millimeters of mercury per hour at time t for $0 \le t \le 4$. Give a formula for the barometric pressure for $0 \le t \le 4$.

28. A woman's investments are worth \$1,000,000 at the beginning of 1999 and are decreasing at the rate of $800t^{1/3}$ dollars per year t years after the beginning of 1999 until the beginning of 2002. Give a formula for the value of her investments as a function of time.

EXPLORATION:

Perform the integration in Exercises 29 through 38.

29.[O] $\displaystyle\int (x+3)^2\,dx$

30.[A] $\displaystyle\int_2^1 \frac{1-w}{w^3}\,dw$

31. $\displaystyle\int_1^3 \left(1+\frac{3}{x^2}\right)^2 dx$

32. $\displaystyle\int_0^1 (1+3t^2)^2\,dt$

33. $\displaystyle\int \frac{x-x^3}{\sqrt{x}}\,dx$

34.[O] $\displaystyle\int_{-8}^{9} P(t)\,dt$ where $P(t)$ equals $t^{1/3}$ for $t < 0$ and equals $t^{1/2}$ for $t > 0$.

35.[A] $\displaystyle\int_0^3 G(x)\,dx$ where $G(x)$ equals x for $x < 2$ and equals $3x^2$ for $x \ge 2$.

36. $\displaystyle\int_{-30}^{30} H(x)\,dx$ where $H(x)$ equals 100 for $x < 0$ and equals -100 for $x > 0$.

37. $\displaystyle\int_0^2 S(t)\,dt$ where $S(t)$ equals $5t^4$ for $t < 1$ and equals $3t^{-4}$ for $t > 1$

38. What is the value of $\displaystyle\int_0^2 (ax^4 - 20x)\,dx$ if the minimum of $y = ax^4 - 20x$ occurs at $x = 1$?

39. Evaluate $\displaystyle\int_{-5}^{5} \left|x^2 - 9\right|\,dx$.

40.[A] Give formulas for the antiderivative $y = V(x)$ of $y = x^{-2}$, defined for $x \ne 0$, such that $V(1) = 2$ and $V(-1) = 3$.

41. Give formulas for the function $y = W(x)$, continuous for all x, such that $W(0) = 3, W'(x) = 2x$ for $x < 0$, and $W'(x) = -1$ for $x > 0$.

42.[A] What is $Z(15) - Z(-15)$ if $y = Z'(x)$ is an odd function that is continuous for all x?

43.[A] What fraction of the area of the rectangle $0 \leq x \leq 9, 0 \leq y \leq 3$ is occupied by the region between $y = \sqrt{x}$ and the x-axis for $0 \leq x \leq 9$?

44.[A] A ball that is rolling down a grass-covered hill has velocity $v(t) = 5 + \frac{3}{2}\sqrt{t}$ feet per second t seconds after it was thrown. **(a)** What is its initial velocity (its velocity at $t = 0$)? **(b)** When is its velocity 11 feet per second? **(c)** How far does it go in 16 seconds?

45. Air is pumped into a balloon at the rate of $r = 3\sqrt{t}$ cubic feet per minute from time $t = 0$ (minutes) until it breaks. Its volume is one cubic foot at $t = 0$ and it breaks when its volume is 17 cubic feet. When does it break?

46. Water flows into a tank at the rate of $10t^4 + 100t + 10$ cubic feet per day at time t (days) and leaks out at the constant rate of 6 cubic feet per day. The tank contains 25 cubic feet of water at $t = 0$. How much does it contain a day later?

47.[A] A balloon's upward velocity is $v = 9t - t^3$ feet per minute t minutes after it is released from six feet above the ground. How high is it above the ground when its upward acceleration is -3 feet per second^2?t

48. A small airplane is flying north with the constant air speed of 125 miles per hour. The wind is blowing south at the speed of t^3 miles per hour at time t (hours) for $0 \leq t \leq 6$. Describe **(a)** its motion and **(b)** its position relative to the ground for $0 \leq t \leq 6$.

49.[O] A ball is rolling in a straight line toward the west on a horizontal sidewalk. Its velocity toward the west is $v = t^3 - 4t$ feet per second at time t (seconds) for $-3 \leq t \leq 1$. **(a)** Where is it at $t = 1$ relative to its position at $t = -3$? **(b)** What is the total distance it travels from $t = -3$ to $t = 1$?

50.[A] What is the total distance an object travels between times $t = 0$ and $t = 5$ (minutes) if its velocity is $v = 3t - t^2$ meters per minute at time t?

51. Find the constant k such that the region between $y = \sqrt[3]{x}$ and the x-axis for $0 \leq x \leq k$ has area 60.75.

(End of Section 6.5)