Section 6.7

(3/20/08)

Integrals involving transcendental functions

In this section we derive integration formulas from formulas for derivatives of logarithms, exponential functions, hyperbolic functions, and trigonometric functions.

Topics:

- ***Integrals of*** $\mathbf{y = x^{-1}}$
- ***Integrals of exponential functions***
- ***Integrals of the hyperbolic sine and cosine functions***
- ***Integrals involving trigonometric functions***
- ***Integrals of*** $\mathbf{y = 1/\sqrt{a^2 - x^2}}$ ***and*** $\mathbf{y = 1/(a^2 + x^2)}$

Integrals of $\mathbf{y = x^{-1}}$

The integration formula

$$\int x^n \, dx = \frac{1}{n+1} \, x^{n+1} + C$$

from Section 6.5 does not give integrals of $x^{-1} = 1/x$ because we cannot set $n = -1$ in the fraction $1/(n+1)$. Instead, we use an integration formula obtained from the formula for the derivative of the natural logarithm,

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \quad \text{for } x > 0. \tag{1}$$

Theorem 1 *For x in any interval not containing zero,*

$$\int \frac{1}{x} \, dx = \ln|x| + C. \tag{2}$$

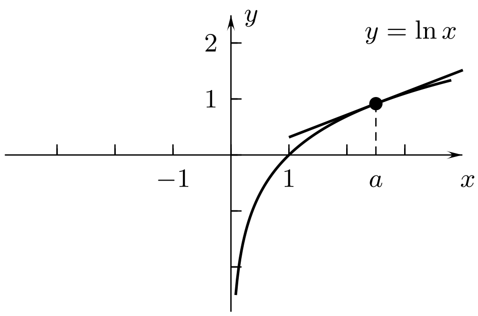
Proof: The differentiation formula **(1)** translates into the integration formula

$$\int \frac{1}{x} \, dx = \ln x + C \quad \text{for } x > 0$$

which is **(2)** for positive x. This formula states that $y = \ln x$ is an antiderivative of $y = 1/x$ for positive x.

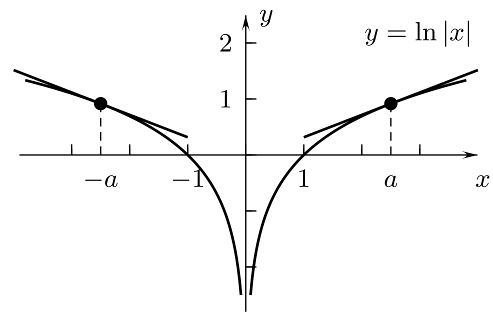
What are the antiderivatives of $y = 1/x$ for negative x, where $\ln x$ is not defined? To answer this, we look at the graph of $y = \ln x$ and its tangent line at $x = a$ for a positive constant a, as in Figure 1. The slope of the tangent line to $y = \ln x$ at $x = a$ is $1/a$ by **(1)**. The graph of an antiderivative of $y = 1/x$ for negative x has to be such that its tangent line at $x = -a$ has slope $1/(-a) = -1/a$.

We can obtain the graph of such an antiderivative for $x < 0$ by taking the mirror image about the y-axis of the graph of $y = \ln x$, as in Figure 2, so that the tangent line at $x = -a$ is the mirror image of the tangent line at $x = a$. This makes the slope of the tangent line at $x = -a$ be the negative $-1/a$ of the slope of the tangent line at $x = a$ since taking the mirror image of a line multiplies the run between two points on the line by -1, without changing the rise. The slope is multiplied by -1.



Tangent line of slope $\frac{1}{a}$

FIGURE 1



Tangent lines of slopes $\frac{1}{-a}$ and $\frac{1}{a}$

FIGURE 2

The function that equals $\ln x$ for $x > 0$ and whose graph for $x < 0$ is the mirror image of the graph of $y = \ln x$ is $y = \ln|x|$. Consequently,

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad \text{for all } x \neq 0 \tag{3}$$

and this gives **(2)**.

We could also derive **(3)** for negative x by using **(1)** and the Chain Rule. For $x < 0$, $|x| = -x$ and

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln(-x)) = \frac{1}{-x}\frac{d}{dx}(-x) = \frac{1}{x}. \quad \textbf{QED}$$

Example 1 Give the exact and approximate decimal values of $\int_2^5 \frac{1}{x}\,dx$ and $\int_{-5}^{-2} \frac{1}{x}\,dx$.

SOLUTION Formula **(2)** with the Fundamental Theorem gives

$$\begin{aligned}
\int_2^5 \frac{1}{x}\,dx &= \left[\int \frac{1}{x}\,dx\right]_2^5 = \Big[\ln|x|\Big]_2^5 \\
&= \ln|5| - \ln|2| = \ln(5) - \ln(2) \doteq 0.916291 \\
\int_{-5}^{-2} \frac{1}{x}\,dx &= \left[\int \frac{1}{x}\,dx\right]_{-5}^{-2} = \Big[\ln|x|\Big]_{-5}^{-2} \\
&= \ln|-2| - \ln|-5| = \ln(2) - \ln(5) \doteq -0.916291. \;\Box
\end{aligned}$$

Example 2 Imagine you had \$50,000 in investments at the beginning of the year 2001 and that their value was increasing at the rate of $R = 800/t$ dollars per year t years after the beginning of 2000 for $t \geq 1$. What would have been the value of your investments at the beginnings of 2002 and 2003? Give the answers rounded to the nearest cent.

SOLUTION Let $I(t)$ be the value of the investments t years after the beginning of 2000. Then $dI/dt = 800/t$ for $t \geq 1$, so that by **(2)**

$$I(t) = \int \frac{800}{t}\, dt = 800 \ln t + C.$$

We write $\ln t$ instead of $\ln |t|$ in this formula since t is positive. Setting $t = 1$ and $I = 50{,}000$ gives $50{,}000 = 800 \ln(1) + C$. Since $\ln(1) = 0$, the constant is $C = 50{,}000$ and $I(t) = 800 \ln t + 50{,}000$. At the beginning of 2002, the investments would have been worth $I(2) = 800 \ln(2) + 50{,}000 \doteq \$50{,}554.52$. At the beginning of 2003 they would have been worth $I(3) = 800 \ln(3) + 50{,}000 \doteq \$50{,}878.89$. □

Integrals of exponential functions

The differentiation formulas from Section 3.3,

$$\frac{d}{dx}(e^x) = e^x \tag{4}$$

$$\frac{d}{dx}(b^x) = \ln(b) b^x \tag{5}$$

which are valid for all x and any positive constant b yield the next theorem.

Theorem 2 *For all x and any positive constant $\neq 1$,*

$$\int e^x\, dx = e^x + C \tag{6}$$

$$\int b^x\, dx = \frac{1}{\ln(b)} b^x + C. \tag{7}$$

Proof: Formula **(6)** is a direct consequence of **(4)**, and **(8)** can be obtained from **(5)** by dividing both sides by $\ln(b)$. **QED**

Example 3 Find a formula for the function $y = g(x)$ such that $g'(x) = e^x$ for all x and $g(2) = 10$.

SOLUTION We obtain $g(x) = \int e^x\, dx = e^x + C$ from **(4)**. Then setting $x = 2$ and $g(2) = 10$ yields $10 = e^2 + C$, so that $C = 10 - e^2$ and $g(x) = e^x + 10 - e^2$. □

We use **(4)** to evaluate a definite integral in the next example.

Example 4 Find the area of the region between the x-axis and the curve $y = x^2 + e^x$ for $-1 \le x \le 1$.

SOLUTION Since $y = x^2 + e^x$ is positive for $-1 \le x \le 1$, the area is

$$\int_{-1}^{1} (x^2 + e^x)\, dx = \left[\int (x^2 + e^x)\, dx\right]_{-1}^{3} = \left[\tfrac{1}{3}x^3 + e^x\right]_{-1}^{1} = \left[\tfrac{1}{3}(1^3) + e^1\right] - \left[\tfrac{1}{3}(-1)^3 + e^{-1}\right]$$
$$= [\tfrac{1}{3} + e] - [-\tfrac{1}{3} + e^{-1}] = \tfrac{2}{3} + e - e^{-1}.$$

We used here the integration formula $\int x^2\ dx = \frac{1}{3}x^3 + C$ from Section 6.5 and Theorem 2 above. The region is shown in Figure 3. □

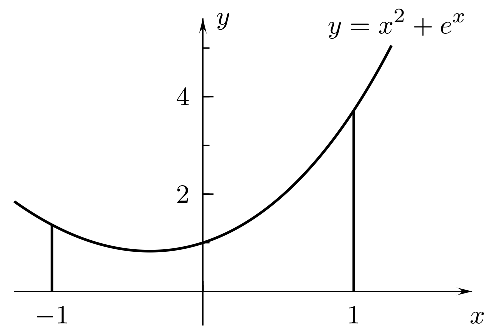


FIGURE 3

Example 5 Evaluate $\int_0^2 3^x\ dx$.

SOLUTION Formula **(7)** with $b = 3$ gives

$$\int_0^2 3^x\ dx = \left[\int 3^x\ dx\right]_0^2 = \left[\frac{1}{\ln(3)} 3^x\right]_0^2 = \frac{1}{\ln(3)}(3^2 - 3^0) = \frac{8}{\ln(3)}. \ \square$$

Integrals of the hyperbolic cosine and sine functions

The next theorem gives formulas for integrating the hyperbolic functions $y = \sinh x$ and $y = \cosh x$.

Theorem 3 *For any x,*

$$\int \cosh x\ dx = \sinh x + C \qquad \textbf{(8)}$$

$$\int \sinh x\ dx = \cosh x + C. \qquad \textbf{(9)}$$

Proof: Formulas **(8)** and **(9)** are consequences of the differentiation formulas,

$$\frac{d}{dx}(\sinh x) = \cosh x \quad \text{and} \quad \frac{d}{dx}(\cosh x) = \sinh x$$

from Section 3.2, which are valid for all x. **QED**

Example 6 Evaluate $\int_0^k (2\cosh x + 5\sinh x)\,dx$, where k is an arbitrary constant.

Solution Formulas **(8)** and **(9)** yield

$$\begin{aligned}\int_0^k (2\cosh x + 5\sinh x)\,dx &= \left[\int (2\cosh x + 5\sinh x)\,dx\right]_0^k = \Big[2\sinh x + 5\cosh x\Big]_0^k \\ &= [2\sinh(k) + 5\cosh(k)] - [2\sinh(0) + 5\cosh(0)] \\ &= 2\sinh(k) + 5\cosh(k) - 5\end{aligned}$$

since $\sinh(0) = 0$ and $\cosh(0) = 1$. □

Integrals involving trigonometric functions

Formulas for the derivatives of the trigonometric functions translate into the next theorem.

Theorem 4 (Some integrals of trigonometric functions) *For any x where the functions are defined,*

$$\int \cos x\,dx = \sin x + C \tag{10}$$

$$\int \sin x\,dx = -\cos x + C \tag{11}$$

$$\int \sec^2 x\,dx = \tan x + C \tag{12}$$

$$\int \csc^2 x\,dx = -\cot x + C \tag{13}$$

$$\int \sec x \tan x\,dx = \sec x + C \tag{14}$$

$$\int \csc x \cot x\,dx = -\csc x + C \tag{15}$$

Proof: Formulas **(10)**, **(12)**, and **(14)** follow directly from the differentiation formulas, $\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\tan x = \sec^2 x$, and $\frac{d}{dx}\sec x = \sec x \tan x$. Formulas **(11)**, **(13)**, and **(15)** can be obtained by multiplying the formulas $\frac{d}{dx}\cos x = -\sin x$, $\frac{d}{dx}\cot x = -\csc^2 x$, and $\frac{d}{dx}\csc x = -\csc x \cot x$ by -1. **QED**

Example 7 A car is 30 miles north of a town at time $t = 0$ (hours) and its velocity toward the north is $v(t) = 60 + 5\cos t + 8\sin t$ miles per hour for $0 \le t \le 3$. Where is it at $t = 3$? Give the exact and approximate decimal values.

SOLUTION Let $s(t)$ be the car's distance north of the town at time t. Then $s'(t) = v(t) = 60 + 5\cos t + 8\sin t$, so that by **(11)** and **(12)**,

$$s(t) = \int (60 + 5\cos t + 8\sin t)\, dt = 60t + 5\sin t - 8\cos t + C.$$

This formula gives $s(0) = 60(0) + 5\sin(0) - 8\cos(0) = -8 + C$. On the other hand, $s(0) = 30$ from the statement of the problem, so that, $-8 + C = 30, C = 38$ and $s(t) = 60t + 5\sin t - 8\cos t + 38$. We set $t = 3$ to obtain

$$\begin{aligned} s(3) &= 60(3) + 5\sin(3) - 8\cos(3) + 38 \\ &= 5\sin(3) - 8\cos(3) + 218. \end{aligned}$$

The car is $5\sin(3) - 8\cos(3) + 218 \doteq 226.63$ miles north of the town at $t = 3$. □

Example 8 What is the area of the region between $y = \sec^2 x$ and the x-axis for $-1 \le x \le 1$?

SOLUTION Since $\sec^2 x > 0$ for $-1 \le x \le 1$, the area is

$$\begin{aligned} \int_{-1}^{1} \sec^2 x\, dx &= \left[\int \sec^2 x\, dx\right]_{-1}^{1} = \Big[\tan x\Big]_{-1}^{1} = \tan(1) - \tan(-1) \\ &= \tan(1) - [-\tan(1)] = 2\tan(1). \end{aligned}$$

The region is drawn in Figure 4. □

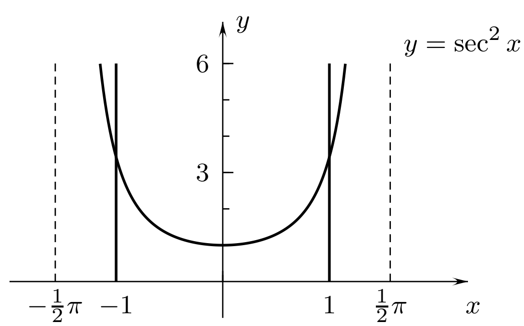


FIGURE 4

Integrals of $\mathbf{y = 1/\sqrt{a^2 - x^2}}$ *and* $\mathbf{y = 1/\sqrt{a^2 + x^2}}$

We use the following differentiation formulas from Section 3.6 to derive the integration formulas in the next theorem.

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}} \tag{16}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2} \tag{17}$$

Theorem 5 *For any positive constant a,*

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C \quad \text{for } -a < x < a \tag{18}$$

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \quad \text{for all } x. \tag{19}$$

Proof: Formula **(16)** with the Chain Rule gives

$$\frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{a}\right)\right] = \frac{1}{\sqrt{1 - (x/a)^2}}\frac{d}{dx}\left(\frac{x}{a}\right)$$
$$= \left(\frac{1}{a}\right)\frac{a}{\sqrt{1 - x^2}} = \frac{1}{\sqrt{a^2 - x^2}}.$$

This yields **(18)**. Similarly, **(17)** gives

$$\frac{d}{dx}\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right] = \frac{1}{a}\left[\frac{1}{1 + (x/a)^2}\right]\frac{d}{dx}\left(\frac{x}{a}\right)$$
$$= \frac{a^2}{a}\left[\frac{1}{a^2 + x^2}\right]\left(\frac{1}{a}\right) = \frac{1}{a^2 + x^2}$$

which establishes **(19)**. **QED**

Example 9 A tank contains 200 gallons of water at time $t = 0$ (hours) and water is drained out at the rate of $100/\sqrt{25 - t^2}$ gallons per hour for $0 \le t \le 4$. How much water is in the tank at $t = 4$? (Give the exact answer and its approximate decimal value.)

SOLUTION Let $V(t)$ be the volume of water in the tank at time t. Since water is draining out of the tank $V'(t) = -\dfrac{100}{\sqrt{25 - t^2}}\, dt$ and

$$V(t) = -\int \frac{100}{\sqrt{25 - t^2}}\, dt = -\int \frac{100}{\sqrt{5^2 - t^2}}\, dt = -100\sin^{-1}\left(\tfrac{1}{5}t\right) + C.$$

Here we used **(17)** with $a = 5$. Set $t = 0$ and $V = 200$: $200 = -100\sin^{-1}\left(\frac{1}{5}(0)\right) + C = C$. This gives
$V(t) = 200 - 100\sin^{-1}\left(\frac{1}{5}t\right)$, so that $V(4) = 200 - 100\sin^{-1}\left(\frac{4}{5}\right)$. At $t = 4$ the tank contains $200 - 100\sin^{-1}\left(\frac{4}{5}\right) \doteq 107.27$ gallons of water. □

Example 10 Find the area of the region between the curve $y = \dfrac{18}{9+x^2}$ and the x-axis for $0 \le x \le 3$.

SOLUTION Since $y = \dfrac{18}{9+x^2}$ is positive the area is given by the integral

$$\int_0^3 \frac{18}{9+x^2}\,dx = \int_0^3 \frac{1}{3^2+x^2}\,dx = \left[\tfrac{1}{3}\tan^{-1}\left(\tfrac{1}{3}x\right)\right]_0^3$$
$$= \tfrac{1}{3}\tan^{-1}(1) - \tfrac{1}{3}\tan^{-1}(0) = \tfrac{1}{3}\left(\tfrac{1}{4}\pi\right) - \tfrac{1}{3}(0) = \tfrac{1}{12}\pi.$$

Here we used **(18)** with $a = 3$ and the facts that $\tan^{-1}(1) = \frac{1}{4}\pi$ and $\tan^{-1}(0) = 0$. The region is shown in Figure 5. □

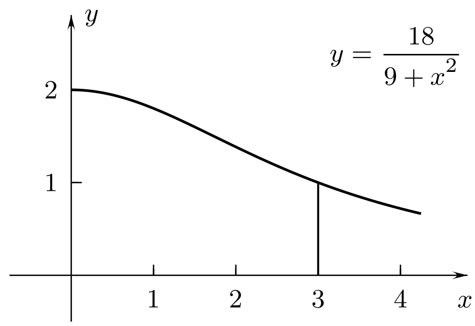


FIGURE 5

Interactive Examples 6.7

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.†

Perform the integration in Examples 1 through 6.

1. $\displaystyle\int \left(\frac{3}{x} + \frac{x}{3}\right) dx$

2. $\displaystyle\int (2x + 5e^x)\,dx$

3 $\displaystyle\int_0^1 (3\cos x - 4\sin x)\,dx.$

4 $\displaystyle\int (2\sec^2 x + 3\csc^2 x)\,dx.$

5. $\displaystyle\int_0^1 \sec x \tan x\,dx$

6. $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$

7. $\displaystyle\int \frac{1}{100+x^2}\,dx$

8. Water is flowing into a tank at the rate of $r = 100/t$ gallons per hour for $t \ge 1$ (hours) and there is 500 gallons in the tank at $t = 1$. How much is in the tank at $t = 10$?

9. What is the area of the region between $y = \sin x + \cos x - 2$ and the x-axis for $0 \le x \le \pi$? Give the exact and approximate decimal answers.

† In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

Exercises 6.7

[A]Answer provided. [O]Outline of solution provided. [C]Graphing calculator or computer required.

CONCEPTS:

1. Why are the integrals in Example 1 at the beginning of the section the negatives of each other?

2. **(a)** Figures 6 through 8 show the rectangles whose areas equal the right Riemann sums for the integrals $\int_1^4 \frac{1}{t}\,dt$, $\int_2^8 \frac{1}{t}\,dt$, and $\int_3^{12} \frac{1}{t}\,dt$ corresponding to the partitions of the intervals of integration into three equal subinterals. Show that the three Riemann sums are equal.
(b) The reasoning in part (a) can be used to show that the Riemann sums for the three integrals for any number of equal subintervals are equal. What does this say about the integrals?

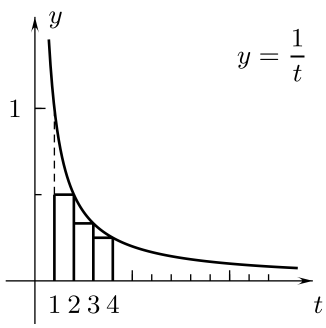


FIGURE 6

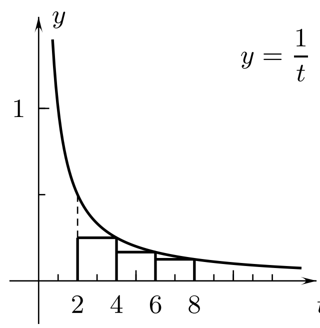


FIGURE 7

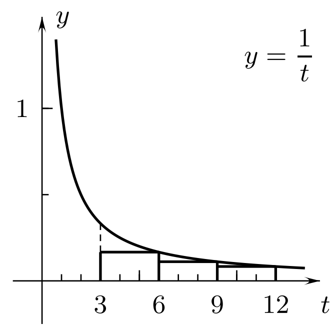
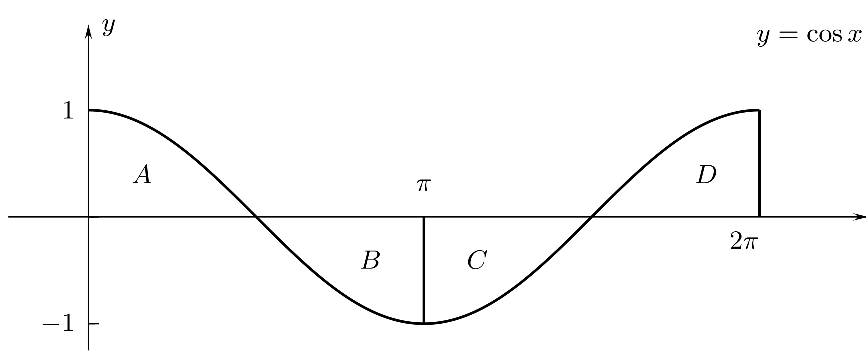


FIGURE 8

3. Exercise 2 gives special cases of the general formula, $\int_1^a \frac{1}{t}\,dt = \int_b^{ab} \frac{1}{t}\,dt$. Use this to show that the function $L(x) = \int_1^x \frac{1}{t}\,dt$ satisfies $L(ab) = L(a) + L(b)$ for any positive numbers a and b. (This is what makes $L(x) = \ln x$ a logarithm.)

4. **(a)** Each of the regions A, B, C, and D in Figure 9, which are set off by the curve $y = \cos t$ and the t-axis, has area 1. Use this fact to describe how $S(x) = \int_0^x \cos t\,dt$ changes as x increases from 0 to 2π. **(b)** Explain the result of part (a) by using results of this section to evaluate the integral $S(t)$.

FIGURE 9



BASICS:

Perform the integration in Exercises 5 through 23.

5.[O] $\int_1^6 \frac{1}{x}\, dx$

6.[O] $\int_{-6}^{-1} \frac{1}{x}\, dx$

7.[O] $\int (x + e^x)\, dx$

8.[O] $\int_0^3 2^x\, dx$

9.[O] $\int (\sin x - 5\cos x)\, dx$

10.[O] $\int_0^{0.5} \sec x \tan x\, dx$

11.[O] $\int \frac{1}{16 + x^2}\, dx$

12.[O] $\int_{-5}^{5} \frac{1}{\sqrt{36 - x^2}}\, dx$

13.[O] $\int_{-4}^{4} e^x\, dx$

14.[O] $\int_0^2 (6x - 2\cos x)\, dx$

15.[A] $\int_{\pi/4}^{3\pi/4} (\sin x + \cos x)\, dx$

16. $\int (\sec x \tan x - \sec^2 x)\, dx$

17. $\int_0^{\pi/3} \sin x\, dx$

18.[O] $\int_{-3}^{3} \frac{1}{\sqrt{25 - x^2}}\, dx$

19.[A] $\int_0^{\sqrt{3}} \frac{1}{1 + x^2}\, dx$

20.[O] $\int_0^{\pi/4} \sec^2 x\, dx$

21. $\int_{\pi/4}^{\pi/3} \cos x\, dx$

22.[O] $\int_{-1}^{1} 10^x\, dx$

23.[A] $\int_1^{e^8} \frac{1}{x}\, dx$

24.[A] $\int_0^{\sqrt{3}} \frac{1}{1 + x^2}\, dx$

25.[O] An object is at the origin on an s-axis and its velocity $v = v(t)$ is zero at time $t = 1$ (seconds). Its acceleration in the positive direction is $a = t^{-2}$ feet per second per second at time t for $t \geq 1$. **(a)** Give a formula for its position $s = s(t)$ (feet) for $t \geq 1$. [C]**(b)** Generate the graph of $a = s(t)$ for $1 \leq t \leq 6$ and copy it on your paper. What are $\lim_{t\to\infty} a(t)$, $\lim_{t\to\infty} v(t)$, and $\lim_{t\to\infty} s(t)$, and how are these facts shown by the graph of $s = s(t)$?

26.[O] Find the area of the region bounded by $y = \sin x + 3$, and the lines $y = 0$, $x = 0$, and $x = 5$.

27.[A] What is the area of the region between $y = (x^2 + 1)^{-1}$ and the x-axis for $-3 \leq x \leq 3$?

28.[O] An object is at $s = -1$ (feet) on an s-axis at $t = 0$ (minutes), and its velocity in the positive s-direction is $v = \sin t$ feet per minute for $t \geq 0$. Find its s-coordinate as a function of $t \geq 0$.

29. A tank contains 10 gallons of water at $t = 0$ (hours) and the rate of flow into the tank is $8t + 5\sin t$ gallons per hour for $t \geq 0$. How much water is in the tank at $t = 5$?

EXPLORATION:

Carry out the integration in Exercises 30 through 32.

30.^A $\displaystyle\int \frac{x^2+1}{x}\,dx$

31.^A $\displaystyle\int_1^5 2^x\,dx$

32. $\displaystyle\int_1^3 (\tfrac{1}{2})^x\,dx$

33.^A Find $a > 0$ so that the area between $y = \dfrac{a}{a^2+x^2}$ and the x-axis for $0 \le x \le 1$ is $\frac{1}{3}\pi$.

34.^A Find the value of the constant b such that the region lying between $y = e^x$ and the x-axis for $0 \le x \le b$ has the same area as the region between $y = 3^x$ and the x-axis for $0 \le x \le 3$.

35. Give formulas for the function $y = F(x)$ such that $F'(x) = 1/x$ for $x \ne 0$, $F(1) = 2$ and $F(-1) = 4$.

36.^A Choose the constant k so that $\displaystyle\int_0^k \frac{1}{\sqrt{4-x^2}}\,dx = \tfrac{1}{6}\pi$.

37. Find k so that $\displaystyle\int_0^k \frac{1}{1+x^2}\,dx = \tfrac{1}{4}\pi$.

38. Find a constant B such that $\displaystyle\int_0^b \frac{1}{1+x^2}\,dx$ differs from $\frac{1}{2}\pi$ by less than 0.01 for $b > B$.

(End of Section 6.7)