Higher dimensional point sets in general position

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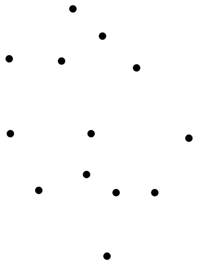
Classical problem of Erdős

$P = N$ points in the plane, no 4 collinear.

Question: What is the size of the largest subset in P in general position (no 3 collinear)?



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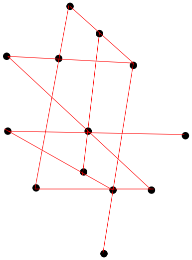
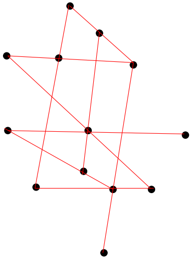


Notation

Let $\alpha_2(N)$ be the largest integer such that every N-element point set in the plane with no 4-collinear members, contains $\alpha_2(N)$ in general position.



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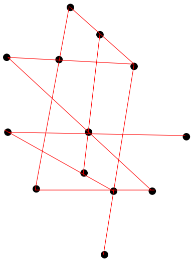

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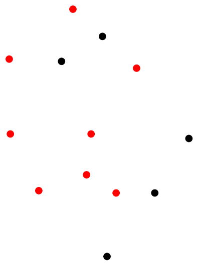

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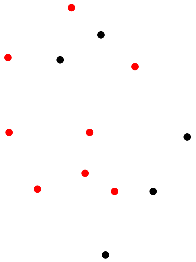

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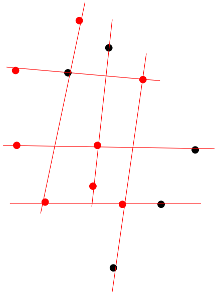

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Let $\alpha_2(N)$ be the largest integer such that every N-element point set in the plane with no 4-collinear members, contains $\alpha_2(N)$ in general position.

Best known lower bound

Theorem (Füredi 1991, Phelps-Rödl 1986)

$\alpha_2(N) \geq \Omega(\sqrt{N \log N})$

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Best known upper bound

Theorem (Balogh-Solymosi 2018)

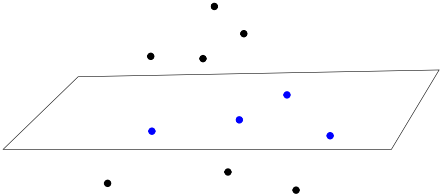
$\alpha_2(N) < N^{5/6+o(1)}$

Higher dimensions

Given:

$P = N$ points in $\mathbb{R}^d$, no $d + 2$ on a hyperplane.

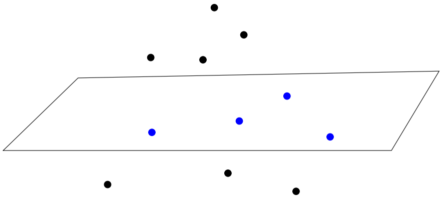
Question: What is the largest subset in general position?



Higher dimensions

Notation

Let $\alpha_d(N)$ be the largest integer such that every N-element point set in $\mathbb{R}^d$ with no $d+2$ on a common hyperplane, contains $\alpha_d(N)$ in general position (no d+1 points on a hyperplane).



Hypergraph

$V = N$ points in $\mathbb{R}^d$

$E = (d+1)$-tuples on a hyperplane

Higher dimensions

Erdős: $\alpha_d(N) \geq \Omega(N^{1/d})$.

Best known lower bound

Theorem (Cardinal, Tóth, Wood 2017, Kostochka, Mubayi, Verstraete 2014)

$\alpha_d(N) \geq \Omega((N \log N)^{1/d})$.

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Density Hales-Jewett

Theorem (Cardinal, Tóth, Wood 2017, Milícceví 2017)

$\alpha_d(N) < o(N)$.

Main result

Theorem (S.-Zeng, 2023)

Let $d \geq 3$. When d is **odd***,*

$$\alpha_d(N) < N^{\frac{1}{2}+\frac{1}{2d}+o(1)}.$$

When d is **even***,*

$$\alpha_d(N) < N^{\frac{1}{2}+\frac{1}{d-1}+o(1)}.$$

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Examples:

- $\alpha_3(N) < N^{2/3+o(1)}$,
- $\alpha_4(N) < N^{3/4+o(1)}$,
- $\alpha_5(N) < N^{3/5+o(1)}$,
- $\alpha_6(N) < N^{7/10+o(1)}$,

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Generalization. No $d+5$ points on a hyperplane, $\alpha_d^*(N) < N^{\frac{1}{2}+o(1)}$.

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Original problem. No $d+2$ points on a hyperplane.
Generalization. No $d+5$ points on a hyperplane,
$\alpha_d^*(N) < N^{\frac{1}{2}+o(1)}$.
Example. $\Omega((N \log N)^{1/3}) < \alpha_3^*(N) < N^{1/2+o(1)}$

Idea of the proof: Subset of $[n]^D$.

Point set
$P = [n]^D$

Hypergraph container method

- Establish a supersaturation result on $[n]^D$.
- Apply the hypergraph container lemma and the probabilistic method (Balogh-Morris-Samotij, Saxton-Thomason 2015).

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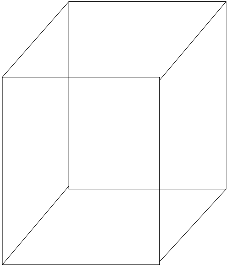
Hypergraph container method

Theorem (Balogh-Solymosi 2018)

Any subset $A \subset [n]^D$ of size $n^{D-\gamma}$, contains at least $cn^{2D-(D+1)\gamma-o(1)}$ collinear triples.

$[n]^D =$ 

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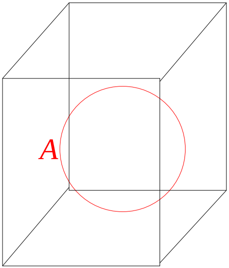

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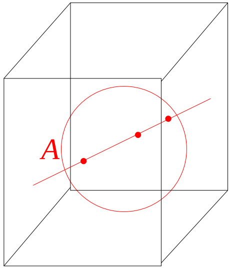
$[n]^D =$ 

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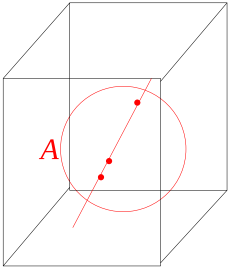
$[n]^D =$ 

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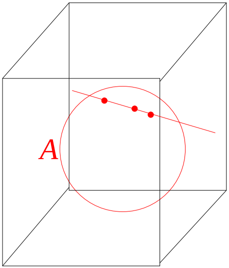

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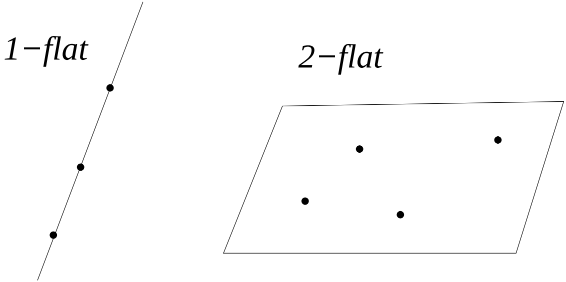
Definition: k-flats. $k = d - 1$

We want many $(k+2)$-tuples on a k-flat.

k-**flat:** k-dimensional affine subspace of $\mathbb{R}^D$.



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Grid:
$P = [n]^D$

We want a supersaturation result for $(k+2)$-tuples on a k-flat

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Adding $k-1$ points to a collinear triple gives $k+2$ points on a k-flat.

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Theorem (Balogh-Solymosi 2018)

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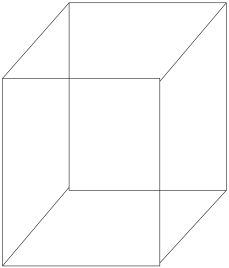
$[n]^D =$ 

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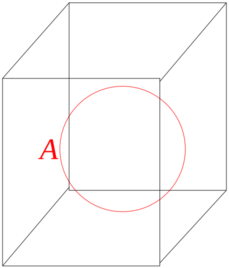

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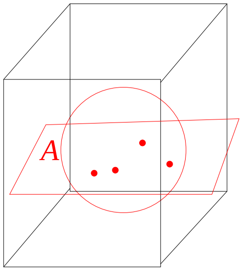

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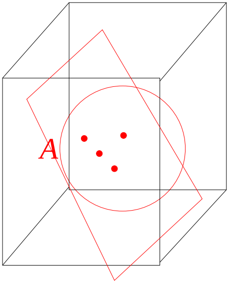
$[n]^D =$ 

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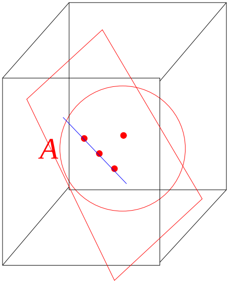
$[n]^D =$ 

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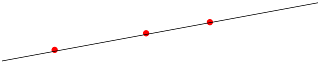
Hypergraph

$V = A \subset [n]^D$ points in $\mathbb{R}^D$

$E = (k+2)$-tuples on a k-flat (**degenerate**)

Large maximum degree $\Rightarrow$ Bad



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- ~~Apply the hypergraph container lemma and the probabilistic method~~.

Degenerate $(k+2)$-tuples

Let A be a set of $k+2$ points in $\mathbb{R}^d$ that lies on k-flat.

Degenerate $(k + 2)$-tuples

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Defintion: We say that A is *degenerate* if there is a proper subset $A' \subset A$ that lies on a $(|A'| - 2)$-flat.

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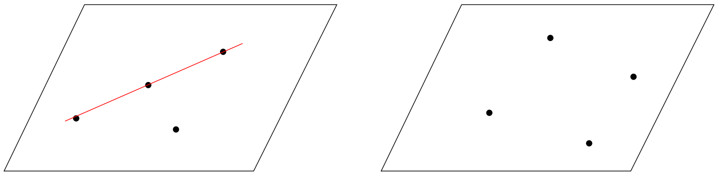
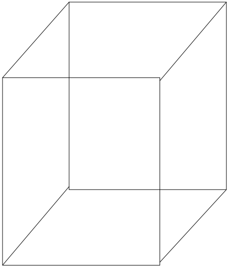
Example: $k = 2$



New supersaturation lemma

Theorem (S.-Zeng 2023)

Let $k \geq 2$ be even. Any subset $A \subset [n]^D$ of size $n^{D-\gamma}$, contains at least $cn^{(k+1)D-(k+2)\gamma}$ non-degenerate $(k+2)$-tuples that lie on a k-flat.

$[n]^D =$ 

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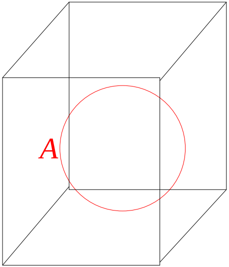

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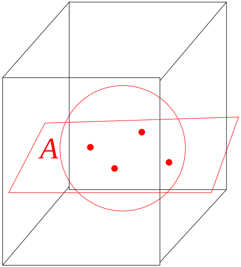
$[n]^D =$ 

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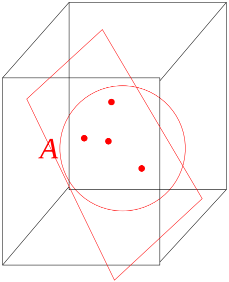

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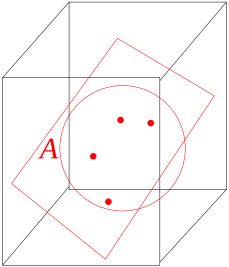
$[n]^D =$ 

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Theorem (Balogh-Solymosi 2018)

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Proof: $A \subset [n]^D$, $|A| = n^{D-\gamma}$. $k \geq 2$ is even.

Set $r = (k+2)/2$

$$A_r = \{a_1 + \cdots + a_r : a_i \in A\} \subset [rn]^D.$$

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Theorem (S.-Zeng 2023)

Let $k \geq 2$ be **even**. *Any subset $A \subset [n]^D$ of size $n^{D-\gamma}$, contains at least $cn^{(k+1)D-(k+2)\gamma}$ non-degenerate $(k+2)$-tuples that lie on a k-flat.*

Proof: $A \subset [n]^D$, $|A| = n^{D-\gamma}$. $k \geq 2$ is even.

Set $r = (k+2)/2$

$$A_r = \{a_1 + \cdots + a_r : a_i \in A\} \subset [rn]^D.$$

$$a_1 + \cdots + a_r = a_1' + \cdots + a_r',$$

$(k+2)$-tuple on a k-flat.

New supersaturation lemma

Proof: $A \subset [n]^D$, $|A| = n^{D-\gamma}$. $k \geq 2$ is even, $r = (k+2)/2$.

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$$\geq (rn)^D \binom{\frac{\sum_v |A_r(v)|}{(rn)^D}}{2} \geq (rn)^D \binom{\frac{|A_r|}{(rn)^D}}{2} \geq \frac{|A_r|^2}{4(rn)^D} \geq cn^{(k+1)D-(k+2)\gamma}$$

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For D large, at least half corresponds to non-degenerate $(k+2)$-tuples on a k-flat.

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Let $d \geq 3$. When d is **odd**,

$$\alpha_d(N) < N^{\frac{1}{2}+\frac{1}{2(d+1)}+o(1)}.$$

When d is **even**,

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Hypergraph container lemma

$\alpha_d(N) < N^{\frac{1}{2}+\frac{1}{2d}+o(1)}$, d is odd.

Set $k = d - 1$, k is even.

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Hypergraph: $H = (V, E)$
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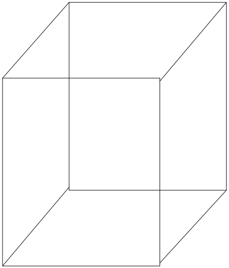
$[n]^D =$ 

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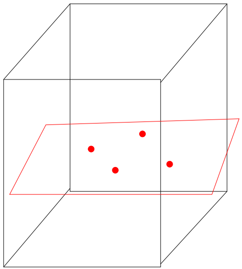
$[n]^D =$ 

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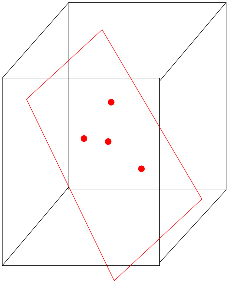
$[n]^D =$ 

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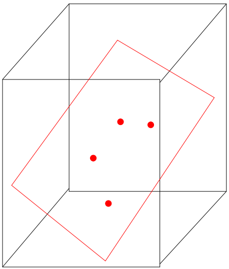
$[n]^D =$ 

Hypergraph container lemma

Theorem (Saxton-Thomason, Balogh-Morris-Samotij 2015)

Given H as above and $\epsilon, \tau \in (0, 1/2)$, if τ is sufficiently small and $\Delta(H, \tau) < c'_k \epsilon$, then there exists a family of containers $\mathcal{C}$ such that

- *Every independent set is in a $C \in \mathcal{C}$,*
- $|\mathcal{C}| < 2^{c|V|\tau \log(1/\epsilon) \log(1/\tau)}$.
- *For each $C \in \mathcal{C}$, $H[C]$ has at most $\epsilon|E|$ edges.*

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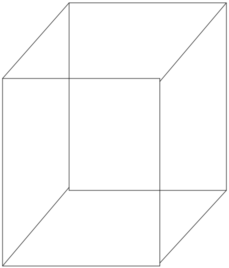

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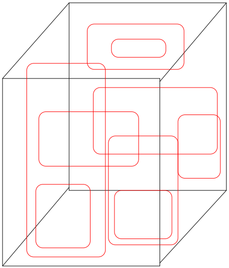

Hypergraph

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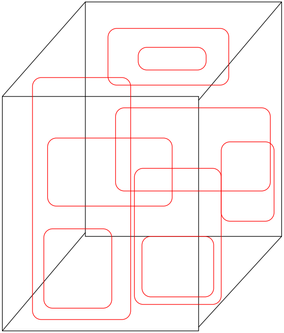

$|\mathcal{C}| < 2^{(c/\alpha)n^{\frac{D}{k+1}+\alpha}\log^2 n}$ $\quad |E[C]| < \epsilon|E|$ $\quad \epsilon = n^{-\alpha}$

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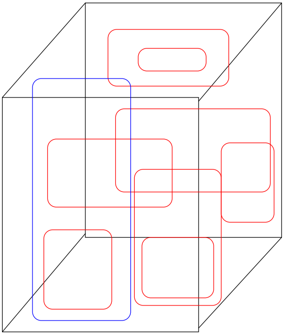

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Supersaturation + Container Lemma

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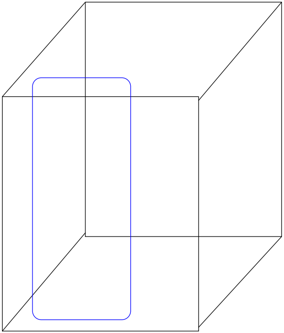

If $|C| > n^{\frac{k}{k+1}D+k}$, repeat the container lemma in $H[C]$.

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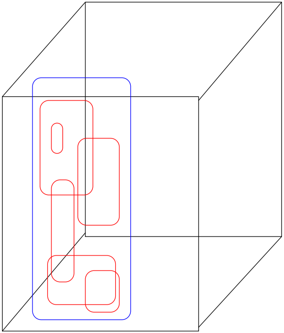

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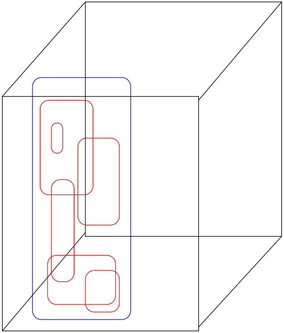
$[n]^D =$ 

$$\mathcal{C}' \leq |\mathcal{C}| 2^{(c/\alpha) n^{\frac{D}{k+1}+\alpha} \log^2 n} \leq 2^{(2c/\alpha) n^{\frac{D}{k+1}+\alpha} \log^2 n}$$

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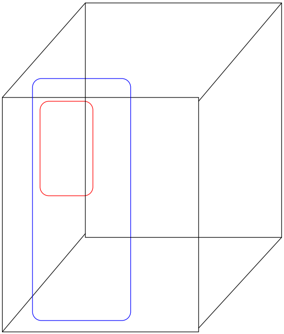
$[n]^D =$ 

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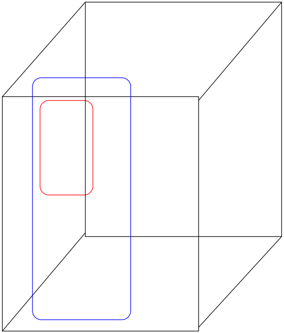

If $|C'| > n^{\frac{k}{k+1}D+k}$, repeat the container lemma in $H[C']$.

After i iterations:

$$|\mathcal{C}^{(i)}| \leq 2^{((i+1)c/\alpha)n^{\frac{D}{k+1}+\alpha}\log^2 n} \qquad |E[C^{(i)}]| < \epsilon^{i+1}|E| = n^{-(i+1)\alpha}|E|.$$

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After $O(kD/\alpha)$ iterations: All containers $|C| < n^{\frac{k}{k+1}D+k}$. Indeed, if $|C| > n^{\frac{k}{k+1}D+k}$

$$\text{Large} < |E[C]| < n^{-10kD}|E|.$$

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Total number of containers:

$$|\mathcal{C}| \leq 2^{(c'/\alpha^2)n^{\frac{D}{k+1}+\alpha}\log^2 n}$$

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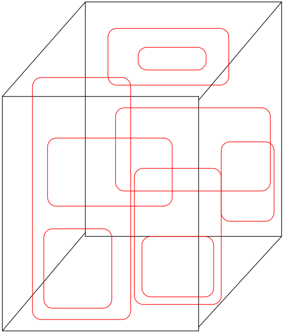

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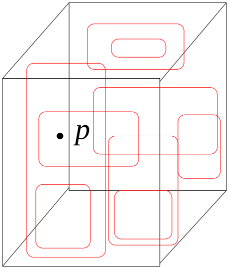
The probabilistic method

Pick a point with probability $p = cn^{-\frac{k}{k+2}(D+2)}$

$[n]^D =$ 

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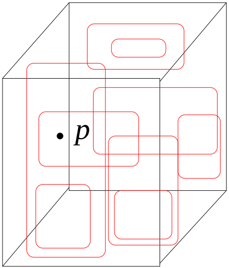
$[n]^D =$ 

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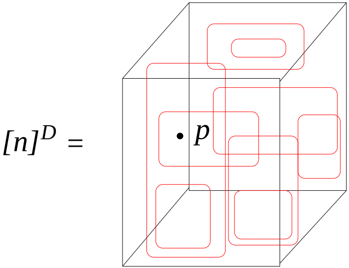

$$\mathbb{E}[\#\text{of points}] = pn^D.$$

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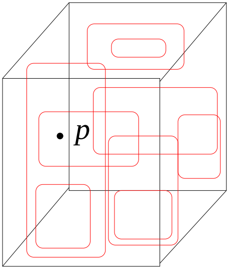
$[n]^D =$ 

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$$\mathbb{E}[\# \text{ of indep sets of size } m = cn^{\frac{D}{2k}+2\alpha}] < |\mathcal{C}| \binom{n^{\frac{k}{k+1}D+k}}{m} p^m = o(1).$$

The probabilistic method

Construction of P in $\mathbb{R}^D$: No $k+3$ points on a k-flat.

$$|P| = \frac{pn^D}{2} = cn^{-\frac{k}{k+2}(D+2)}n^D = cn^{\frac{2(D-k)}{k+2}} = N.$$

$$\alpha(P) < m = cn^{\frac{D}{k+1}+2\alpha} < O(N^{\frac{k+2}{2(k+1)}+\alpha}).$$

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- No $d+2$ on a hyperplane
- Every subset of size $cN^{\frac{1}{2}+\frac{1}{2d}+\alpha}$ contains $d+1$ points on a hyperplane.

Main result

Theorem (S.-Zeng, 2023)

Let $d \geq 3$. When d is **odd**,

$$\alpha_d(N) < N^{\frac{1}{2}+\frac{1}{2d}+o(1)}.$$

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Open problem: Close the gap

$$\alpha_d(N) \geq \Omega((N \log N)^{1/d}).$$

Back to supersaturation: open problem

Theorem (S.-Zeng 2023)

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Question: Can we remove the *even* condition.

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Question: Can we remove the *even* condition.

Theorem (Balogh-Solymosi 2018)

Any subset $A \subset [n]^d$ of size $n^{d-\gamma}$, contains at least $cn^{2d-(d+1)\gamma}$ collinear triples.

Question: Better supersaturation result.

Thank you!