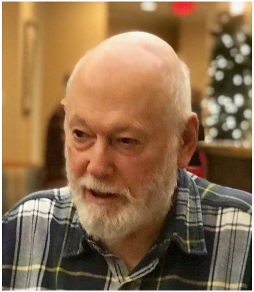
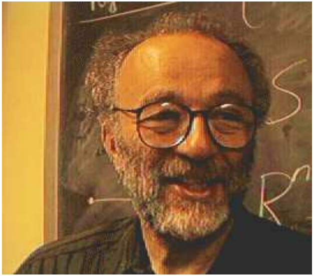
Unavoidable patterns in complete simple topological graphs

Andrew Suk (UC San Diego)

Goodman-Pollack DCG Day, April 12, 2022

Goodman-Pollack DCG Day





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QUASI-PLANAR GRAPHS HAVE A LINEAR NUMBER OF EDGES

PANKAJ K. AGARWAL, BORIS ARONOV, JÁNOS PACH,
RICHARD POLLACK and MICHA SHARIR

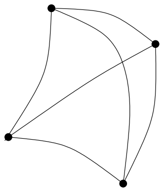
Received January 25, 1996

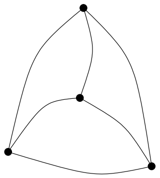
A graph is called *quasi-planar* if it can be drawn in the plane so that no three of its edges are pairwise crossing. It is shown that the maximum number of edges of a quasi-planar graph with n vertices is $O(n)$.

Topological Graph $G = (V, E)$

V = points in the plane.

E = curves connecting the corresponding points (vertices).





Quasi-planar graphs

Theorem (Euler)

Every n-vertex topological graph with no crossing edges has at most $3n - 6$ edges.

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A topological graph is called k-**quasi-planar**, if there are no k pairwise crossing edges.

Conjecture

Every n-vertex k-quasi-planar graph has at most $O_k(n)$ edges.

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- Straight-line edges, $O(n \log n)$ Valtr 1997.

Complete topological graphs

Theorem (Fox-Pach-S. 2022)

Every complete n-vertex topological graph contains n^{ε} pairwise crossing edges.

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Problem

What large patterns can we find in complete topological graphs?

No two disjoint edges

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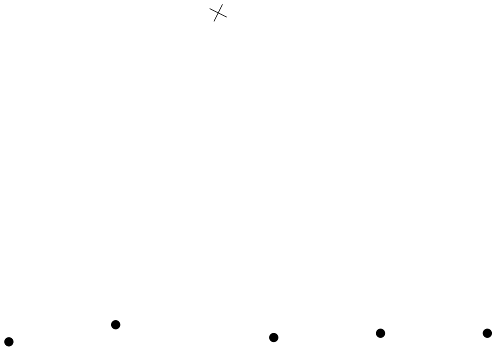

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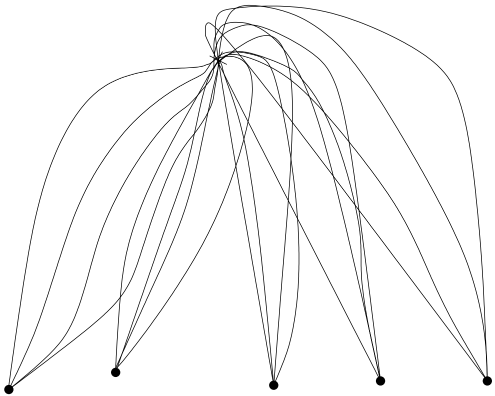

Simple Topological Graph $G = (V, E)$

V = points in the plane.

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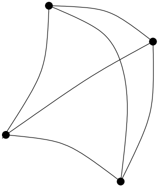
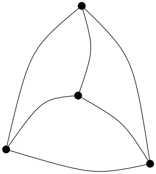
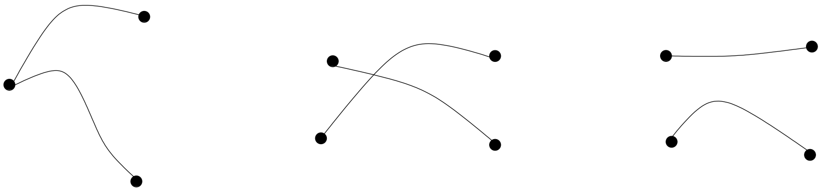
Every pair of edges have at most 1 point in common.





We will only consider simple topological graphs.



Complete simple topological graphs

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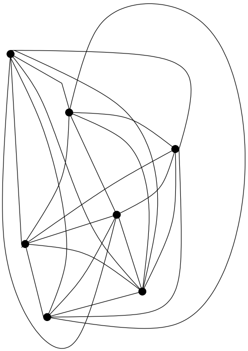
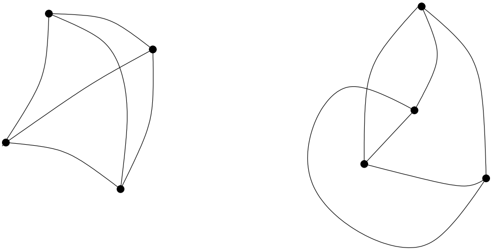


Weakly isomorphic topological graphs

Definition

Topological graphs G and H are **weakly isomorphic** if there is a incidence preserving bijection between $(V(G), E(G))$ and $(V(H), E(H))$ such that two edges in G cross if and only if the corresponding edges in H cross.



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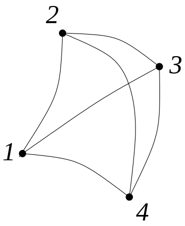

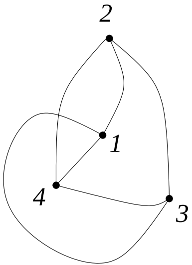

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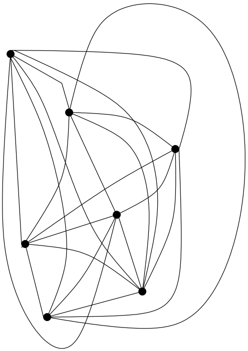
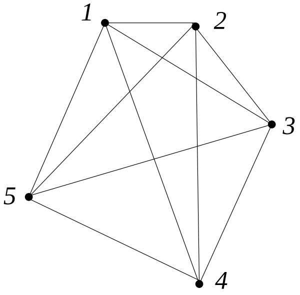


Complete convex (geometric) graph, C_m.

C_5 =



Complete convex (geometric) graph, C_m.

$$V = v_1, \ldots, v_m$$

Edges $v_i v_j$ and $v_k v_\ell$ cross if and only if $i < k < j < \ell$ or $k < i < \ell < j$.

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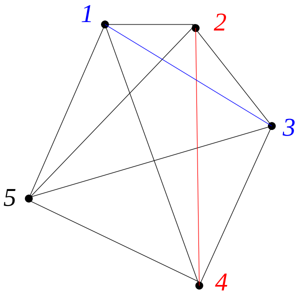

Problem. Does every sufficiently large complete simple topological graph contain a topological subgraph that is weakly isomorhpic to C_5?

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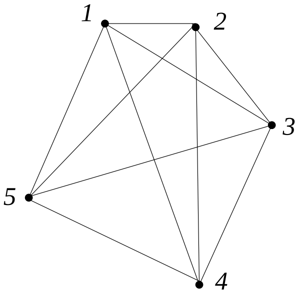

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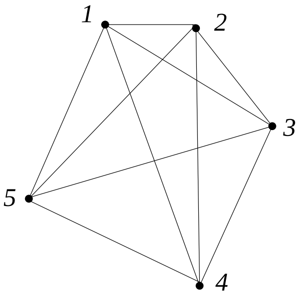
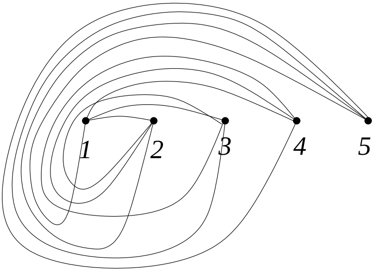


Answer: No!

Twisted complete graph, T_m.

$T_5 =$



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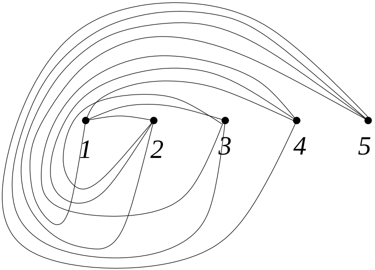

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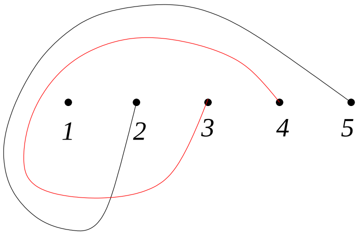

Twisted complete graph

Harborth-Mengersen '92: T_m does not contain a subgraph weakly isomorphic to C_5.

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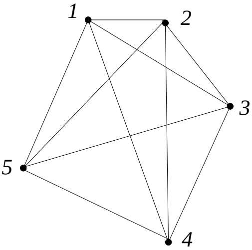

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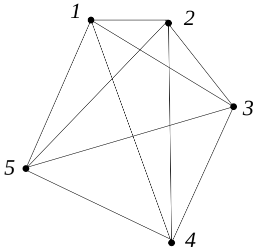

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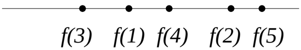

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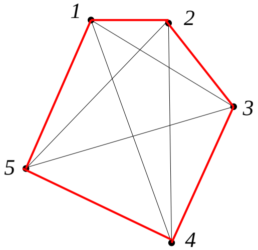

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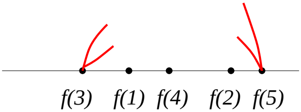

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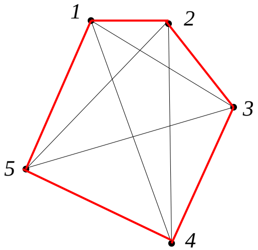

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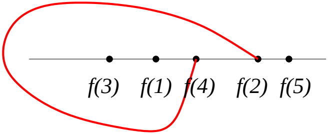

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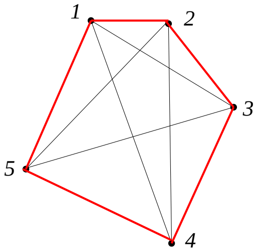

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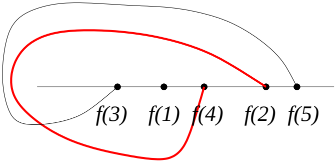

A Ramsey-type theorem

Theorem (Pach-Solymosi-Tóth 2003)

Every complete simple topological graph on n vertices contains a topological subgraph on $m = \Omega(\log^{1/8} n)$ vertices that is weakly isomorphic to C_m or T_m.

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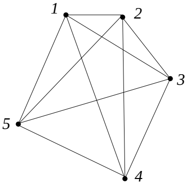

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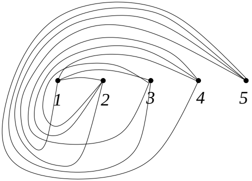

New result

Theorem (S.-Zeng 2022+)

Every complete simple topological graph on n vertices contains a topological subgraph on $m = (\log n)^{1/4-o(1)}$ vertices that is weakly isomorphic to C_m or T_m.

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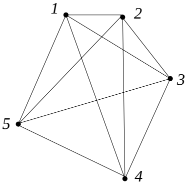

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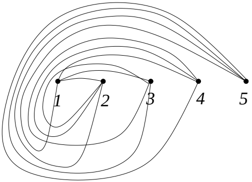

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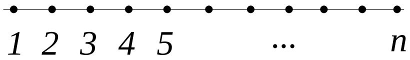
Uniformly at random draw half circles above or below the axis.



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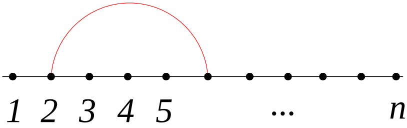

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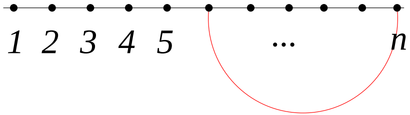

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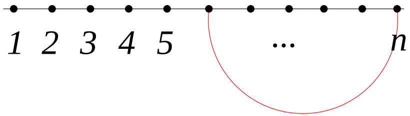
Applying the probabilistic method: There is an n-vertex simple topological graph that does not contain a topological subgraph on $m = \lfloor c \log n \rfloor$ vertices that is weakly isomorphic to C_m or T_m.



Non-crossing path

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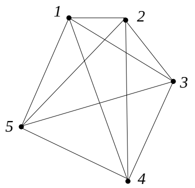

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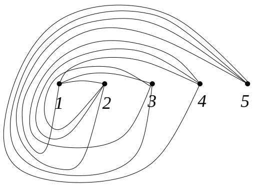

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Conjecture

There is an absolute constant $\epsilon > 0$*, such that every complete n-vertex simple topological graph contains a non-crossing path on* n^{ϵ} *vertices.*

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Pairwise disjoint edges. S., Fulek and Ruiz-Vargas, Ruiz-Vargas: $n^{1/2-o(1)}$ pairwise disjoint edges.

Non-crossing path

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There is an absolute constant $\epsilon > 0$, such that every complete n-vertex simple topological graph contains a non-crossing path on n^{ϵ} vertices.

Problem: Can we find an edge that crosses very few other edges?

Short edge application

Theorem (S.-Zeng 2022+)

Every complete simple topological graph on n vertices contains a topological subgraph on $m = (\log n)^{1/4-o(1)}$ vertices that is weakly isomorphic to C_m or T_m.

Let $h = h(n)$ be the smallest integer such that every complete n-vertex simple topological graph contains an edge crossing at most h other edges.

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Valtr, Kynčl-Valtr: $\Omega(n^{3/2}) < h(n) < O(n^2/\log^{1/4} n)$.

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New bound: $h(n) \leq \frac{n^2}{(\log n)^{1/2-o(1)}}$.

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Conjecture

$$h(n) < n^{2-\epsilon}.$$

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Proof.

$K_n =$

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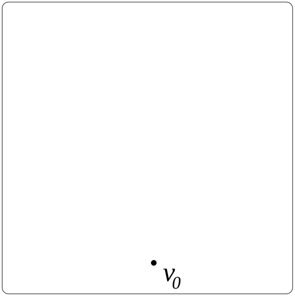

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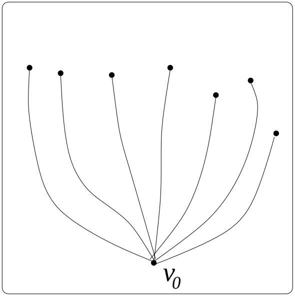
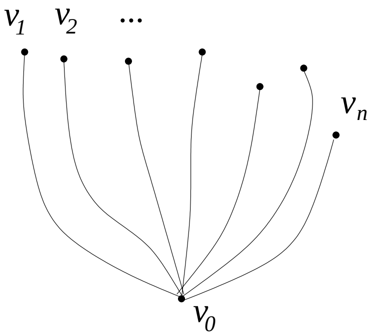
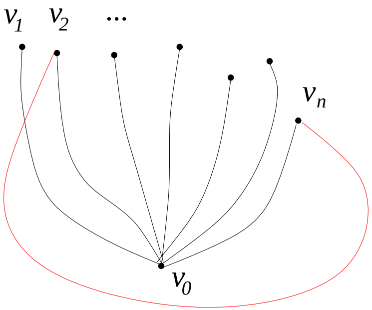


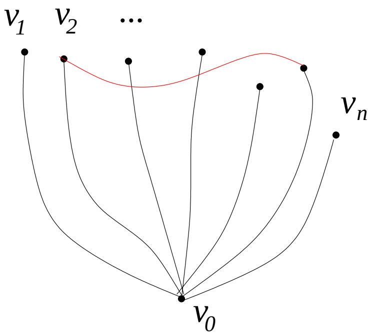
Proof



Proof

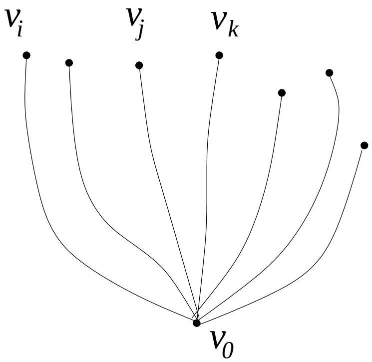


Proof



Observation

Pach-Solymosi-Tóth: For $v_i < v_j < v_k$, we there are only 4 configurations.



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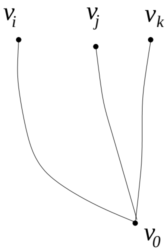

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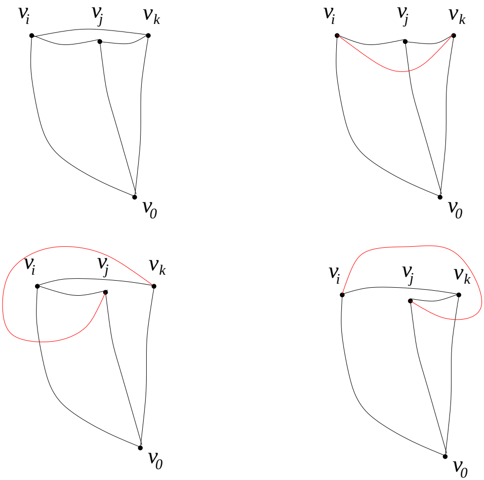
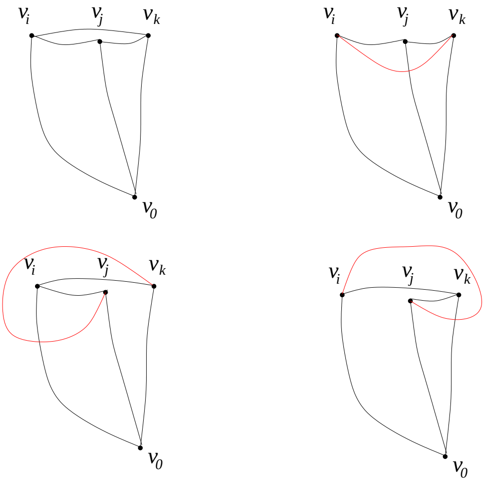


Observation

Pach-Solymosi-Tóth: Color the triple (v_i, v_j, v_k) from $\{000, 001, 010, 100\}$



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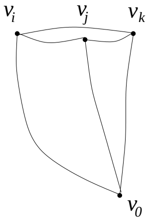
000



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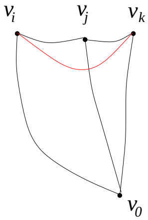
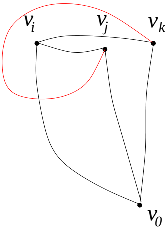


100



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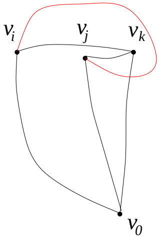
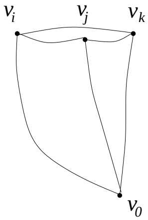


Observation

Goal: Find a monochromatic clique with respect to some color in $\{000, 001, 010, 100\}$.

000



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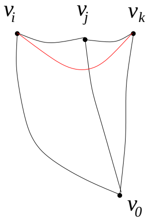

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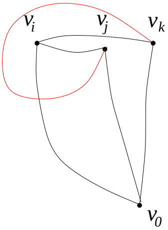

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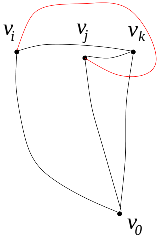


Improvements

Theorem (Pach-Solymosi-Tóth 2003)

Every complete simple topological graph on n vertices contains a topological subgraph on $m = \Omega(\log^{1/8} n)$ vertices that is weakly isomorphic to C_m or T_m.

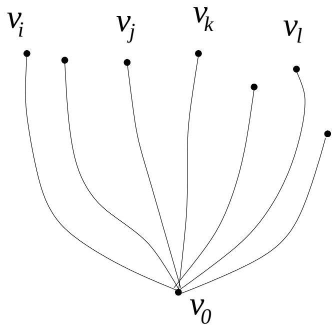
Rough idea:

1. Erdős-Rado greedy argument on the triples.
2. Erdős-Szekeres monotone subsequence theorem.

(Plus some nice topological arguments)

Transitive colors: 001, 100

For $v_i < v_j < v_k < v_\ell$, if (v_i, v_j, v_k) and (v_j, v_k, v_ℓ) have color 001, then so does (v_i, v_j, v_ℓ) and (v_i, v_k, v_ℓ).



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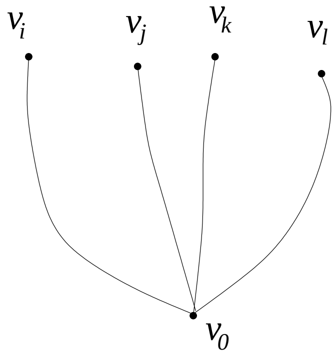

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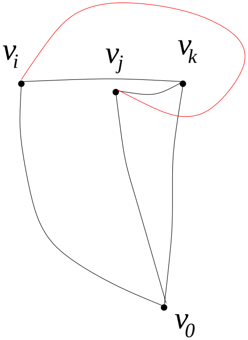

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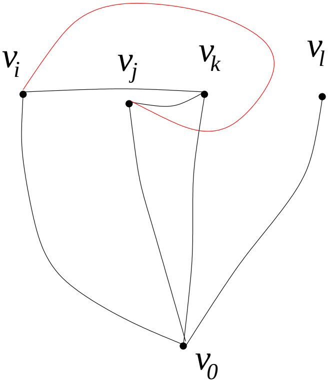

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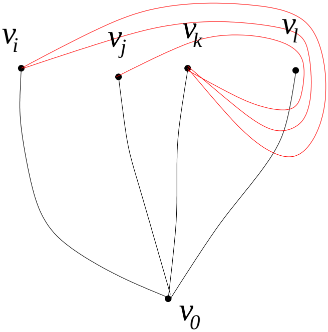

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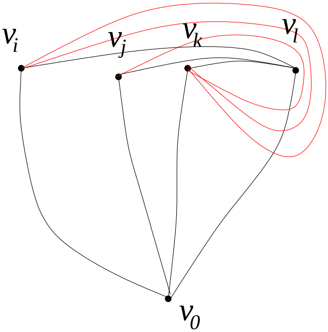
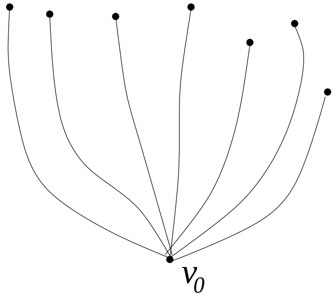


Monotone path

Monotone Path: $u_1 < u_2 < \cdots < u_m$, (u_i, u_{i+1}, u_{i+2})



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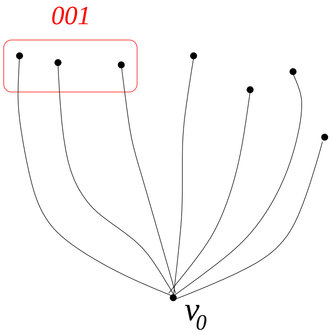

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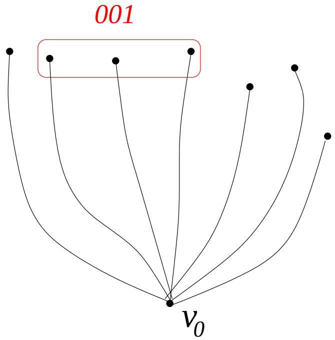

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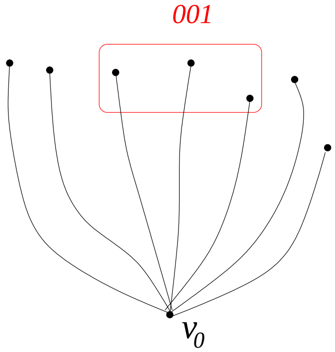

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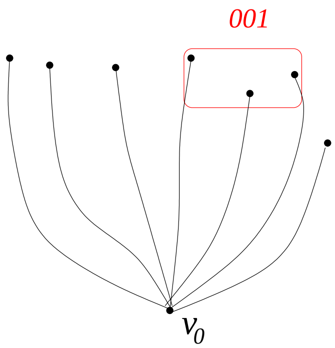

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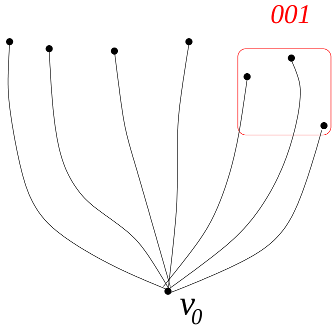

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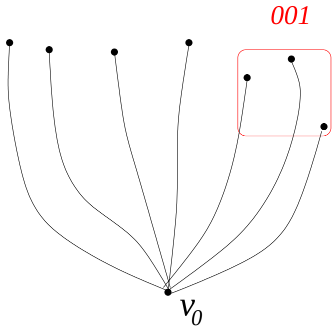


Every triple is 001, we have T_m.

Coloring properties

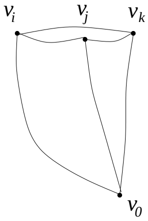
000. Not transitive

100. Transitive

010. Not transitive

001. Transitive

000



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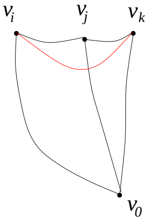

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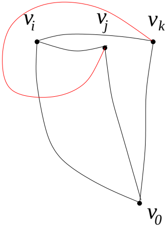

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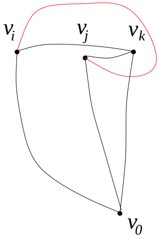

Improvements

Theorem (Pach-Solymosi-Tóth 2003)

Every complete simple topological graph on n vertices contains a topological subgraph on $m = \Omega(\log^{1/8} n)$ vertices that is weakly isomorphic to C_m or T_m.

Rough idea:

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Transitive observation: $m = (\log n)^{1/6 - o(1)}$.

Improvements

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2. ~~Erdős-Szekeres monotone subsequence theorem.~~

Transitive observation: $m = (\log n)^{1/6 - o(1)}$.

Online Ramsey Game: Builder vs. Painter. $m = (\log n)^{1/4 - o(1)}$.

□

Conclusion

Theorem (S.-Zeng 2022+)

Every complete simple topological graph on n vertices contains a topological subgraph on $m = (\log n)^{1/4-o(1)}$ vertices that is weakly isomorphic to C_m or T_m.

Non-trivial construction?

Problem

Find an n-vertex complete simple topological graph with no subgraph on $m = (\log n)^{1-\epsilon}$ vertices that is weakly isomorphic to C_m or T_m.

Non-crossing paths and short edge

Conjecture

There is an absolute constant $\epsilon > 0$, such that every complete n-vertex simple topological graph contains a non-crossing path on n^ϵ vertices.

Conjecture

$$h(n) < n^{2-\epsilon}.$$

Thank you!