Unavoidable patterns in complete simple topological graphs

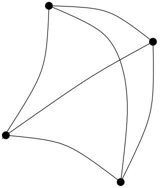
Andrew Suk (UC San Diego)

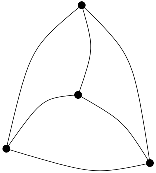
April 26, 2022

Topological Graph $G = (V, E)$

V = points in the plane.

E = curves connecting the corresponding points (vertices).





Quasi-planar graphs

Theorem (Euler)

Every n-vertex topological graph with no crossing edges has at most $3n - 6$ edges.

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Conjecture

Every n-vertex k-quasi-planar graph has at most $O_k(n)$ edges.

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- $k = 4$, Ackerman 2009.
- $k \geq 5$, $n(\frac{c \log n}{\log k})^{2 \log k - 4}$, Fox-Pach-S. 2022.

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Coloring Intersection Graphs:

- Coloring curves that cross a fixed curve, Rok, Walczak
- Outerstring graphs are -bounded, Rok, Walczak
- Triangle-free intersection graphs of line segments with large chromatic number, Pawlik, Kozik, Krawczyk, Lason, Micek, Trotter, Walczak

Complete topological graphs

Theorem (Fox-Pach-S. 2022)

Every complete n-vertex topological graph contains n^{ε} pairwise crossing edges.

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Problem

What large patterns can we find in complete topological graphs?

Complete topological graphs

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Problem

Can we find a large set of pairwise disjoint edges?

No two disjoint edges

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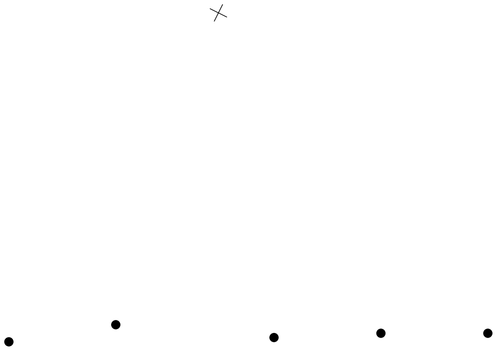

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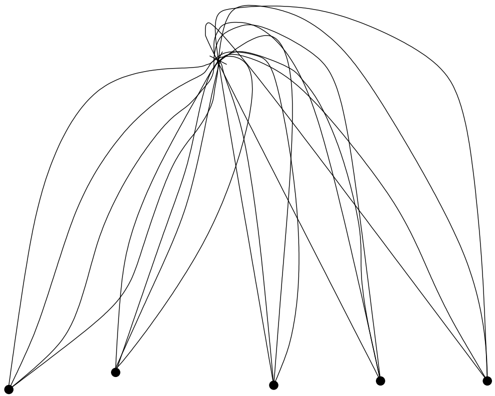

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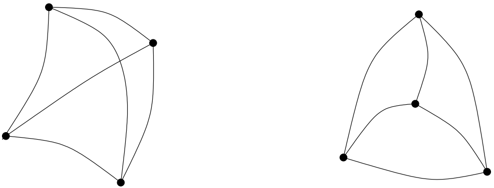
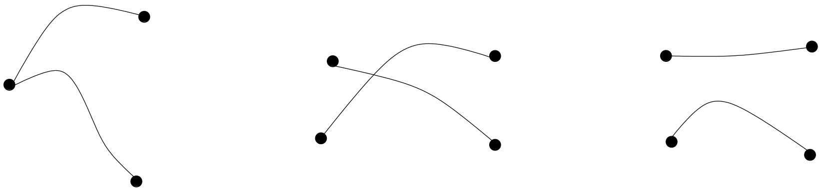


We will only consider simple topological graphs.



Disjoint edges in complete simple topological graphs

Theorem (S. 2013, Fulek-Ruiz-Vargas 2014)

Every complete n-vertex simple topological graph contains $\Omega(n^{1/3})$ *pairwise disjoint edges.*

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New bound: $\Omega(n^{1/2})$, Aichholzer, Garcia, Tejel, Vogtenhuber, Weinberger, 2022.

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Dense simple topological graphs?

Back to the sparse setting: Thrackles

Conjecture (Conway)

Every n-vertex simple topological graph with no 2 disjoint edges has at most n edges.

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Every n-vertex simple topological graph with no 2 disjoint edges has at most $1.3984n$ edges.

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For fixed $k \geq 3$, every n-vertex simple topological graph with no k pairwise disjoint edges has at most $O(n \log^{4k-8} n)$ edges.

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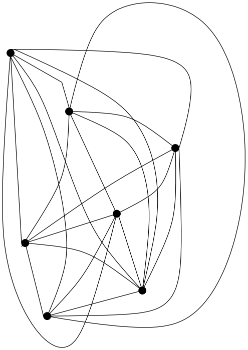
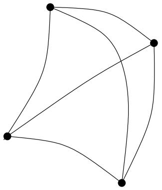
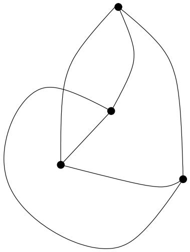


Weakly isomorphic topological graphs

Definition

Topological graphs G and H are **weakly isomorphic** if there is a incidence preserving bijection between $(V(G), E(G))$ and $(V(H), E(H))$ such that two edges in G cross if and only if the corresponding edges in H cross.





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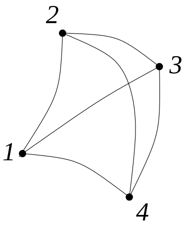

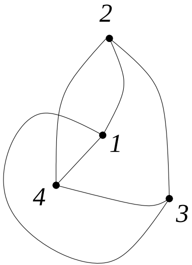

Complete simple topological graphs

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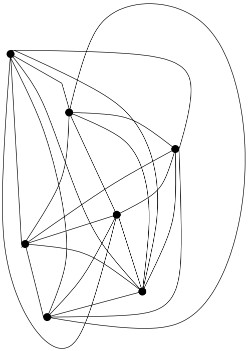
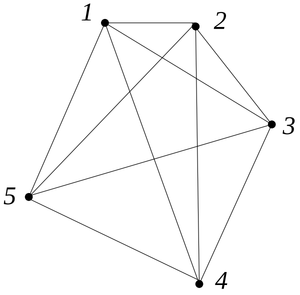


Homogeneous configurations

Complete convex (geometric) graph, C_m.

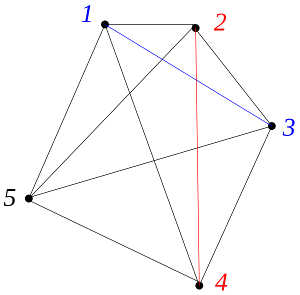
C_5 =



Complete convex (geometric) graph, C_m.

$$V = v_1, \ldots, v_m$$

Edges v_iv_j and v_kv_ℓ cross if and only if $i < k < j < \ell$ or $k < i < \ell < j$.

$C_5 =$ 

Problem. Does every sufficiently large complete simple topological graph contain a topological subgraph that is weakly isomorhpic to C_5?

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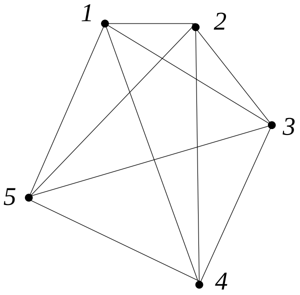

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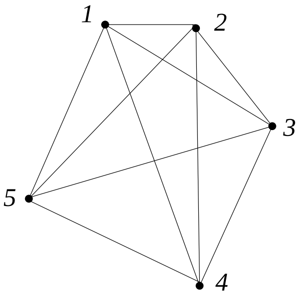

Answer: No!

Twisted complete graph

Twisted complete graph, T_m.

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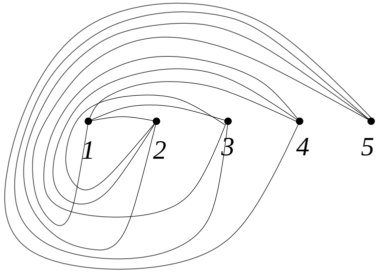

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$$V(T_m) = v_1, \ldots, v_m$$

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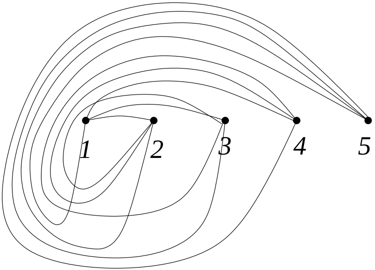

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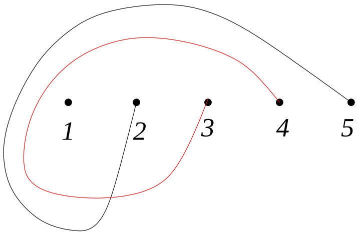
T_5 = 

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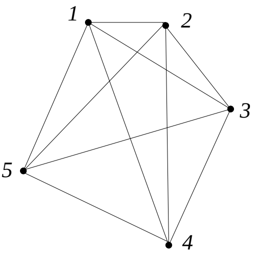

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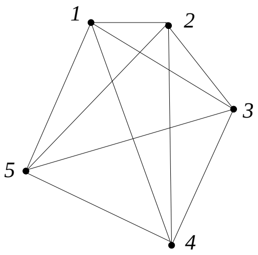

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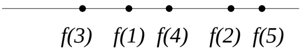

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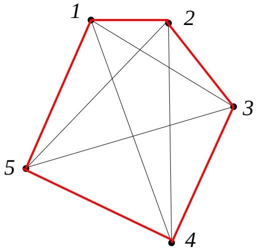

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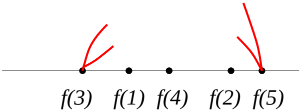

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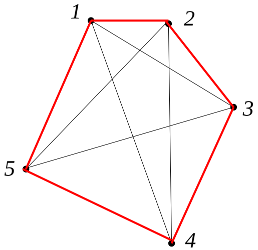

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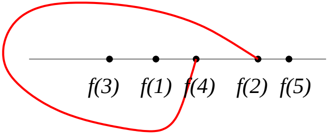

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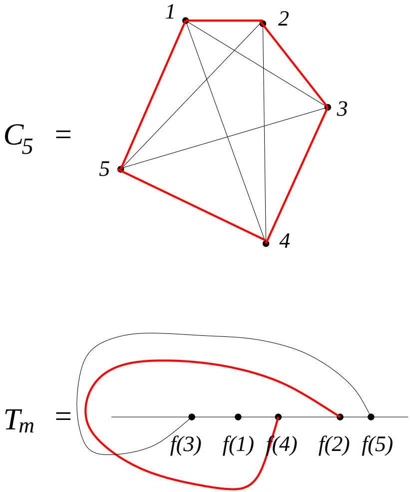

A Ramsey-type theorem

Theorem (Pach-Solymosi-Tóth 2003)

Every complete simple topological graph on n vertices contains a topological subgraph on $m = \Omega(\log^{1/8} n)$ vertices that is weakly isomorphic to C_m or T_m.

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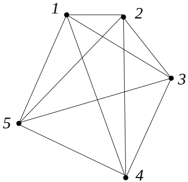

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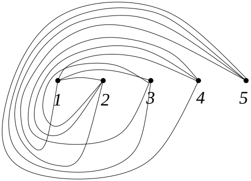

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Theorem (S.-Zeng 2022+)

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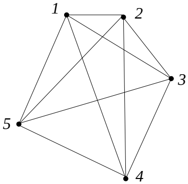

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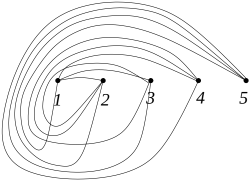

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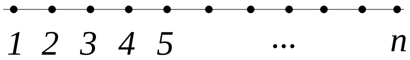

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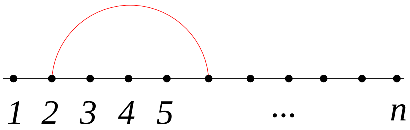

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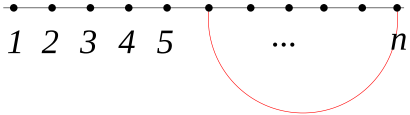

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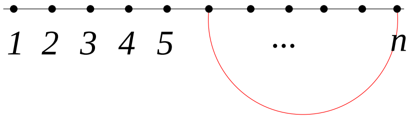
Applying the probabilistic method: There is an n-vertex simple topological graph that does not contain a topological subgraph on $m = \lfloor c \log n \rfloor$ vertices that is weakly isomorphic to C_m or T_m.



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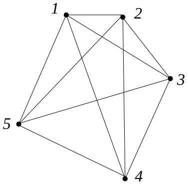

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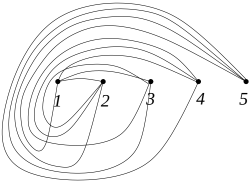

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True for pairwise disjoint edges. S., Fulek, Ruiz-Vargas, Aichholzer, Garcia, Tejel, Vogtenhuber, Weinberger.

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Problem: Can we find an edge that crosses very few other edges?

Short edge application

Theorem (Pach-Solymosi-Tóth 2003)

Every complete simple topological graph on n vertices contains a topological subgraph on $m = \Omega(\log^{1/8} n)$ *vertices that is weakly isomorphic to* C_m *or* T_m*.*

Let $h = h(n)$ be the smallest integer such that every complete n-vertex simple topological graph contains an edge crossing at most h other edges.

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New bound: $h(n) \leq \frac{n^2}{(\log n)^{1/2-o(1)}}$.

Short edge application

Theorem (S.-Zeng 2022+)

Every complete simple topological graph on n vertices contains a topological subgraph on $m = (\log n)^{1/4 - o(1)}$ vertices that is weakly isomorphic to C_m or T_m.

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Conjecture

$$h(n) < n^{2-\epsilon}.$$

New results

Theorem (S.-Zeng 2022+)

Every complete simple topological graph on n vertices contains a topological subgraph on $m = (\log n)^{1/4-o(1)}$ vertices that is weakly isomorphic to C_m or T_m.

Theorem (S.-Zeng 2022+)

Every complete simple topological graph on n vertices contains a non-crossing path on $(\log n)^{1-o(1)}$ vertices.

Theorem (S.-Zeng 2022+)

Every complete simple topological graph on n vertices contains a topological subgraph on $m = (\log n)^{1/4-o(1)}$ vertices that is weakly isomorphic to C_m or T_m.

Proof.

$K_n =$

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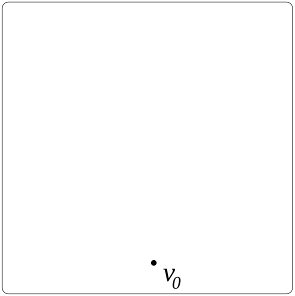

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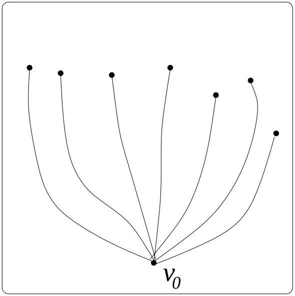
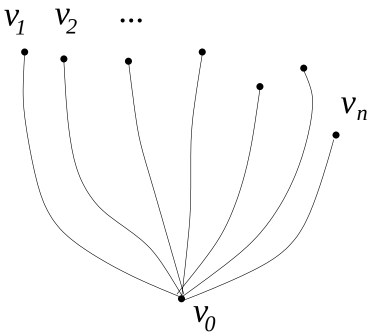


Proof



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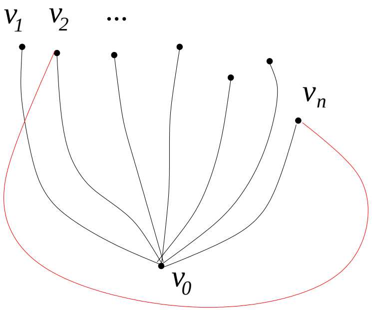

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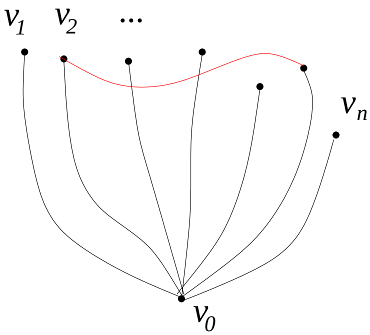
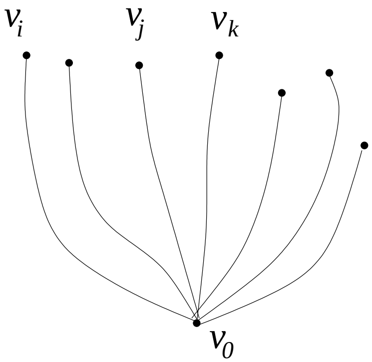


Pach-Solymosi-Tóth: For $v_i < v_j < v_k$, we there are only 4 configurations.



Observation

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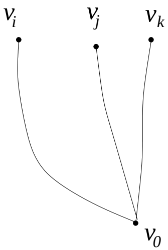

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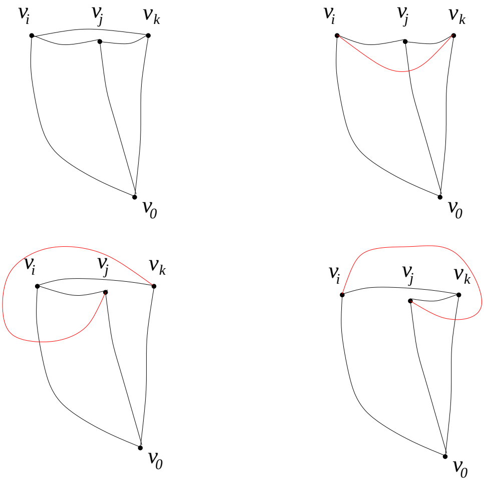
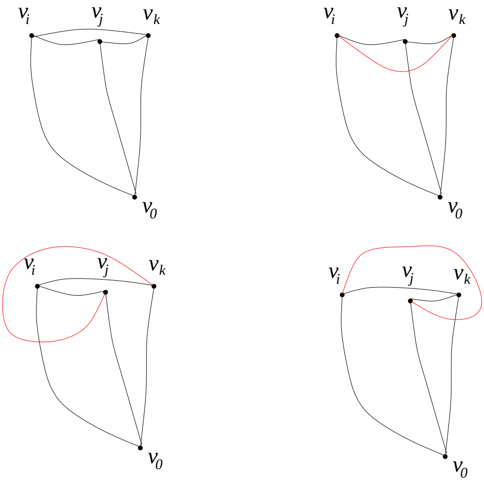


Observation

Pach-Solymosi-Tóth: Color the triple (v_i, v_j, v_k) from $\{000, 001, 010, 100\}$



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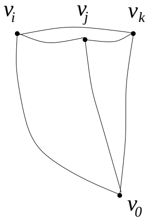
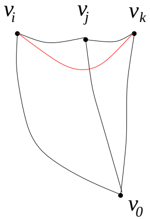
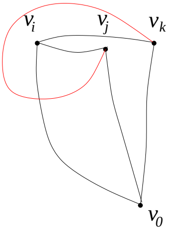
000



010



100



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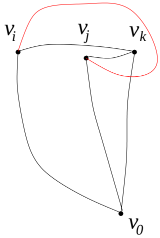
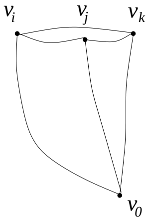


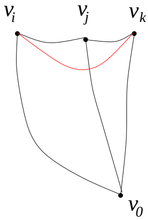
Observation

Goal: Find a monochromatic clique with respect to some color in $\{000, 001, 010, 100\}$.

000



010



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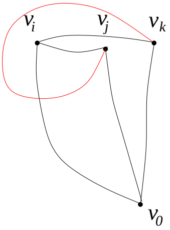
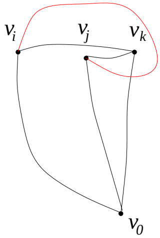


001



Theorem (Pach-Solymosi-Tóth 2003)

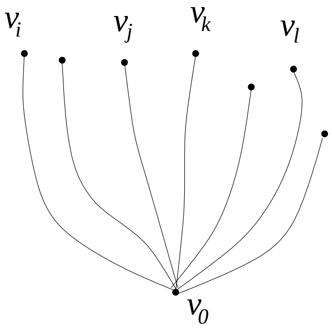
Every complete simple topological graph on n vertices contains a topological subgraph on $m = \Omega(\log^{1/8} n)$ vertices that is weakly isomorphic to C_m or T_m.

Rough idea:

1. Erdős-Rado greedy argument on the triples.
2. Erdős-Szekeres monotone subsequence theorem.

(Plus some nice topological arguments)

For $v_i < v_j < v_k < v_\ell$, if (v_i, v_j, v_k) and (v_j, v_k, v_ℓ) have color 001, then so does (v_i, v_j, v_ℓ) and (v_i, v_k, v_ℓ).



Transitive colors: 001, 100

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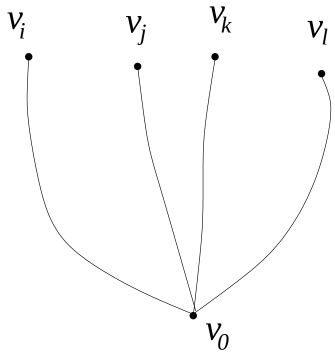

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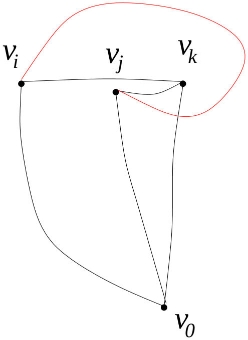

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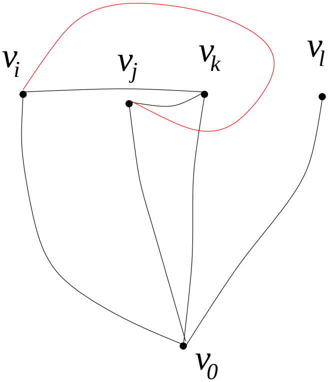

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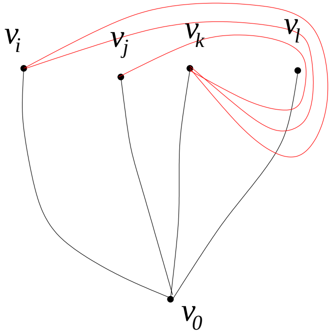

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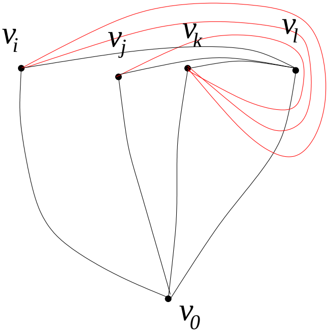
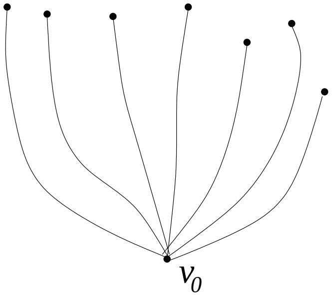


Monotone path

Monotone Path: $u_1 < u_2 < \cdots < u_m$, (u_i, u_{i+1}, u_{i+2})



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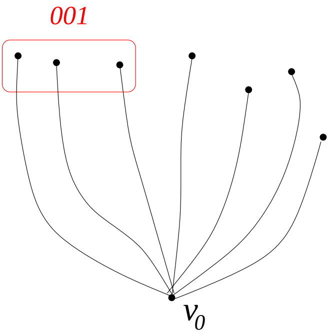

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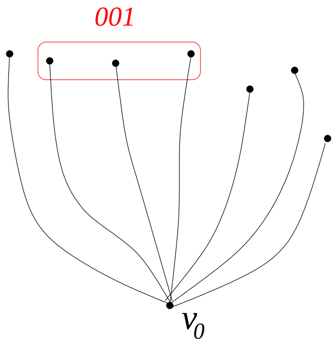

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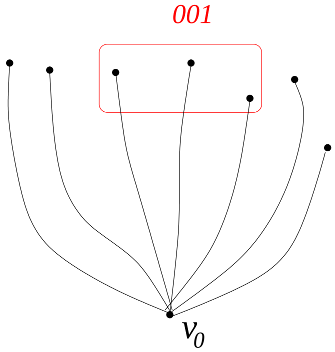

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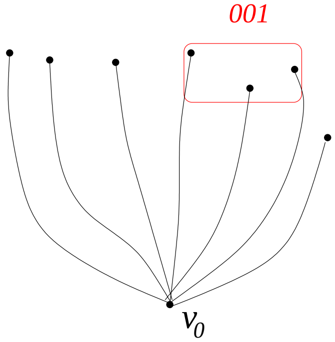

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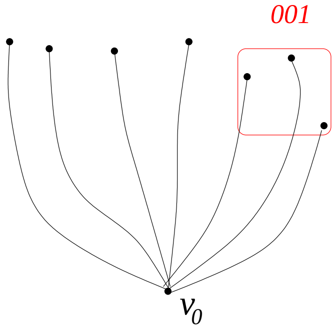

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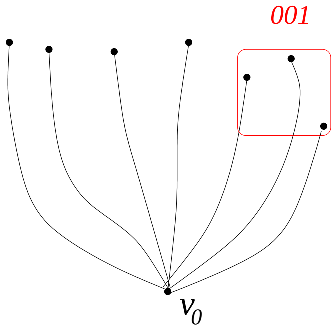


Every triple is 001, we have T_m.

Coloring properties

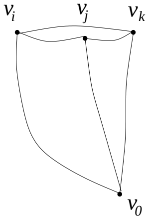
000. Not transitive

100. Transitive

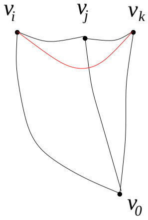
010. Not transitive

001. Transitive

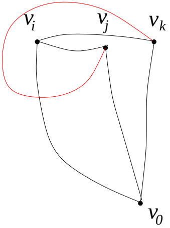
000



010



100



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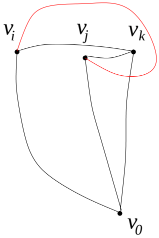

Improvements

Theorem (Pach-Solymosi-Tóth 2003)

Every complete simple topological graph on n vertices contains a topological subgraph on $m = \Omega(\log^{1/8} n)$ vertices that is weakly isomorphic to C_m or T_m.

Rough idea:

1. Erdős-Rado greedy argument on the triples.
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Transitive observation: $m = (\log n)^{1/6 - o(1)}$.

Improvements

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1. Erdős-Rado greedy argument on the triples.
2. ~~Erdős-Szekeres monotone subsequence theorem.~~

Transitive observation: $m = (\log n)^{1/6-o(1)}$.

Online Ramsey Game: Builder vs. Painter. $m = (\log n)^{1/4-o(1)}$.

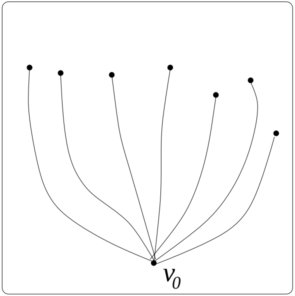
□

Non-crossing path

Theorem (S.-Zeng 2022+)

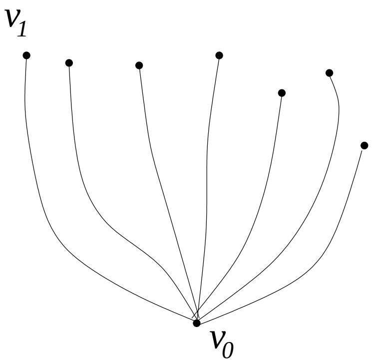
Every complete simple topological graph on n vertices contains a non-crossing path of length $(\log n)^{1-o(1)}$.

Proof.

$K_n =$ 

Non-crossing path

Set $m = \log^2 n$.



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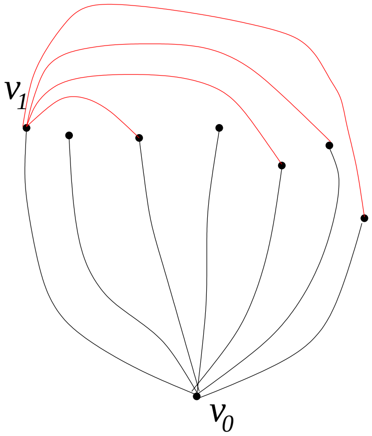
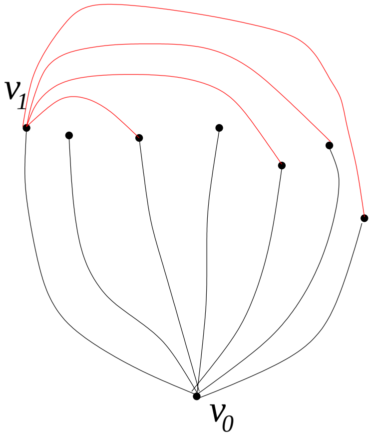


Non-crossing path

Set $m = \log^2 n$. **Case 1.** Planar $K_{2,m}$



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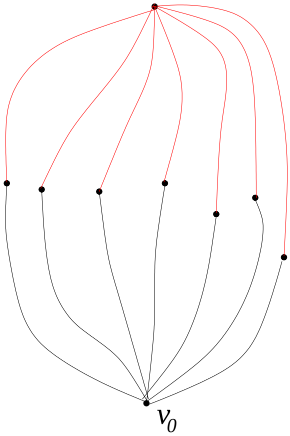

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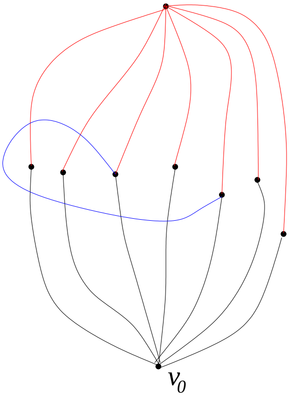

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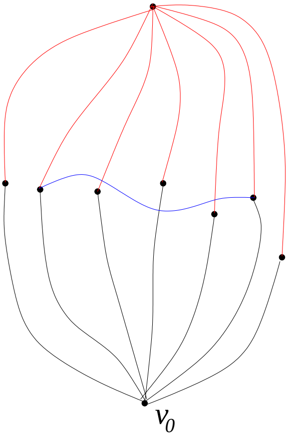

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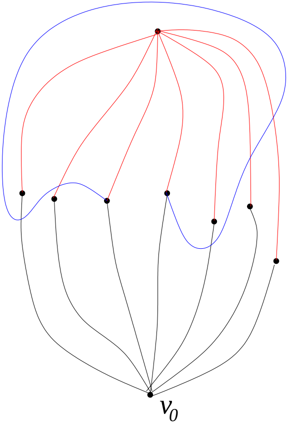

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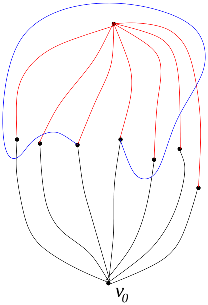
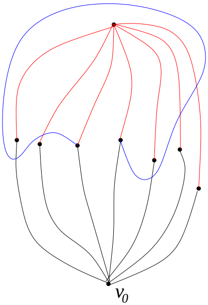


Lemma (Fulek-Ruiz-Vargas 2015)

There is a dense topological subgraph on m vertices that is weakly isomorphic to an x-monotone simple topological graph.

Non-crossing path

Set $m = \log^2 n$. **Case 1.** Planar $K_{2,m}$

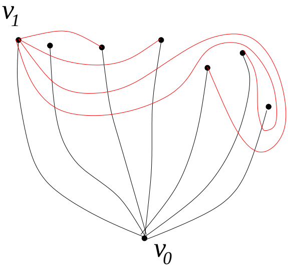


Lemma (Tóth 2000)

Every dense m-vertex simple topological graph with edges drawn as x-monotone curves contains a non-crossing path on $\sqrt{m}$ vertices.

Non-crossing path

Set $m = \log^2 n$. **Case 2.** Decreasing sequence of length n/m



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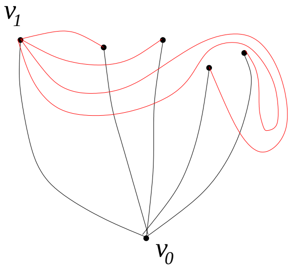

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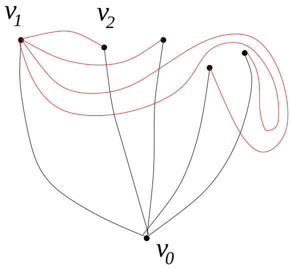
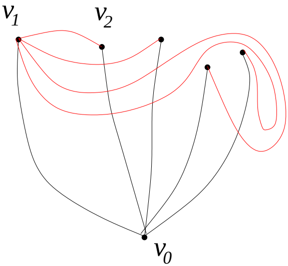


Non-crossing path

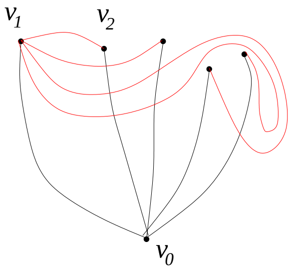
Set $m = \log^2 n$. **Case 2.** Decreasing sequence of length $n/(2m)$



Only keep the vertices inside or outside the triangle $v_0 v_1 v_2$.

Non-crossing path

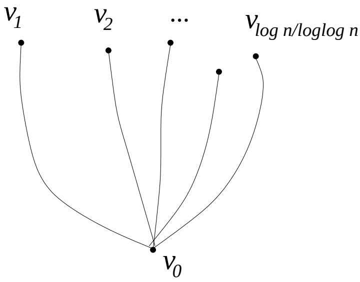
Set $m = \log^2 n$. **Case 2.** Decreasing sequence of length $n/(2m)$



Repeat this process $\log n / \log \log n = (\log n)^{1-o(1)}$ times.

Non-crossing path

Set $m = \log^2 n$. **Case 2.** Decreasing sequence of length $n/(2m)$



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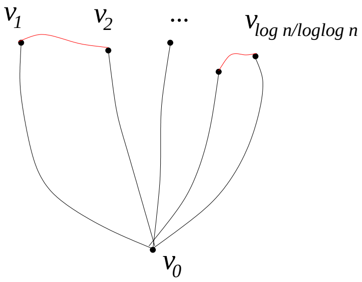
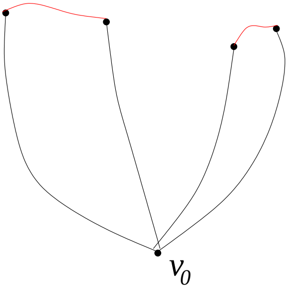


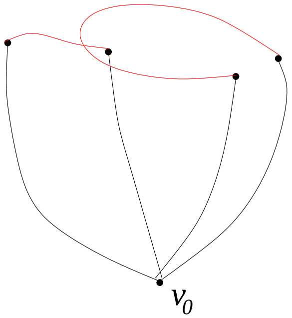
Non-crossing path

Set $m = \log^2 n$. **Case 2.** Decreasing sequence of length $n/(2m)$



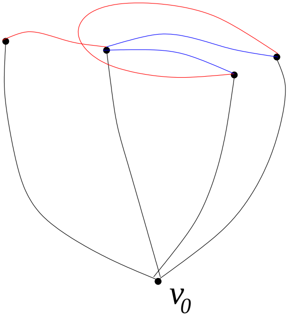
Non-crossing path

Set $m = \log^2 n$. **Case 2.** Decreasing sequence of length $n/(2m)$



Non-crossing path

Set $m = \log^2 n$. **Case 2.** Decreasing sequence of length $n/(2m)$



$\square$

Conclusion

Theorem (S.-Zeng 2022+)

Every complete simple topological graph on n vertices contains a topological subgraph on $m = (\log n)^{1/4-o(1)}$ vertices that is weakly isomorphic to C_m or T_m.

Non-trivial construction?

Problem

Find an n-vertex complete simple topological graph with no subgraph on $m = (\log n)^{1-\epsilon}$ vertices that is weakly isomorphic to C_m or T_m.

Non-crossing paths and short edge

Conjecture

There is an absolute constant $\epsilon > 0$, such that every complete n-vertex simple topological graph contains a non-crossing path on n^{ϵ} vertices.

Conjecture

$$h(n) < n^{2-\epsilon}.$$

Thank you!