On the Erdős-Szekeres convex polygon problem

Andrew Suk

May 25, 2016
Problem (Esther Klein 1933)

*Given an integer $n$, is there a minimal integer $ES(n)$, such that any set of at least $ES(n)$ points in the plane in general position, contains $n$ members in convex position?*
Problem (Esther Klein 1933)

Given an integer $n$, is there a minimal integer $ES(n)$, such that any set of at least $ES(n)$ points in the plane in general position, contains $n$ members in convex position?
Theorem (Erdős-Szekeres 1935, 1960)

\[ 2^{n-2} + 1 \leq ES(n) \leq \binom{2n-4}{n-2} + 1 = O\left(4^n/\sqrt{n}\right). \]

**Conjecture:** \( ES(n) = 2^{n-2} + 1 \)
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Klein (1933): $ES(4) = 5$

Makai (1935): $ES(5) = 9$


For $n \geq 7$, $ES(n)$ is still unknown.
Towards the conjecture $ES(n) = 2^{n-2} + 1$

1935, Erdős-Szekeres: $\binom{2n-4}{n-2} + 1$

1998, Chung-Graham: $\binom{2n-4}{n-2}$

1998, Kleitman-Pachter: $\binom{2n-4}{n-2} - 2n + 7$

1998, Tóth-Valtr: $\binom{2n-5}{n-2} + 2$

2005, Tóth-Valtr: $\binom{2n-5}{n-2} + 1$

2015, Vlachos: $\limsup_{n \to \infty} \frac{ES(n)}{\binom{2n-5}{n-2}} \leq \frac{29}{32}$.

2015, Norin-Yuditsky and Mojarrad-Vlachos: $\limsup_{n \to \infty} \frac{ES(n)}{\binom{2n-5}{n-2}} \leq \frac{7}{8}$.

$= 4^{n-o(n)}$. 
The original upper bound

$$ES(n) \leq \binom{2n - 4}{n - 2} + 1.$$ 

Theorem (Cups-Caps Theorem)

Let $f(k, \ell)$ be the smallest integer $N$ such that any $N$-element point set in the plane in general position contains either a $k$-cup or an $\ell$-cap. Then

$$f(k, \ell) = \binom{k + \ell - 4}{k - 2} + 1.$$
Cups-caps construction

\[ f(k, \ell) \geq f(k - 1, \ell) + f(k, \ell - 1) - 1 \]
Cups-caps construction

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Cups-caps construction

\[ f(k, \ell) \geq f(k - 1, \ell) + f(k, \ell - 1) - 1 \]
Union of an \((n/2)\)-cup and an \((n/2)\)-cap. Note that:
\[
f\left(\frac{n}{2}, \frac{n}{2}\right) = 2^{n-o(n)}
\]

**Question:** Can we (somehow) combine the cups and caps from the cup-cap theorem?
Theorem (S. 2016)

For $n \geq n_0$, where $n_0$ is a large absolute constant

$$ES(n) \leq 2^n + 2n^{3/4}.$$ 

Conjecture: $ES(n) = 2^{n-2} + 1$

Erdős offered $500 for a proof (Graham offered $1000).$
Theorem (Aronov-Erdős-Goddard-Kleitman-Klugerman-Pach-Schulman 1991)

Let $P$ be an $N$-element planar point set in general position. Then there are subsets $A, B \subset P$ such that $|A|, |B| \geq \sqrt{N}$ and $A$ and $B$ are mutually avoiding.
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\[ \sqrt{N} = f(n, n/2) \approx (2.6)^n \implies N \approx (6.75)^n. \]
mutually avoiding sets

**Theorem (Aronov-Erdős-Goddard-Kleitman-Klugerman-Pach-Schulman 1991)**

Let \( P \) be an \( N \)-element planar point set in general position. Then there are subsets \( A, B \subset P \) such that \( |A|, |B| \geq \sqrt{N} \) and \( A \) and \( B \) are mutually avoiding.

**Theorem (Valtr 1994)**

Any point set \( P \) with \( |P| = N \) and with ratio \( c\sqrt{N} \), contains no pair of mutually avoiding sets of size more than \( c'\sqrt{N} \).
Grid-like point sets contain large cups and caps

Theorem (Valtr 1994)

Any point set $P$ with $|P| = N$ and with ratio $c\sqrt{N}$, contains $\Omega(N^{1/3})$ points in convex position.

Basic Idea: Grid like $\Rightarrow$ large cups and caps
Not grid like $\Rightarrow$ large mutually avoiding sets.
Theorem (S. 2016)

For \( n \geq n_0 \), where \( n_0 \) is a large absolute constant

\[ ES(n) = 2^n + o(n). \]

Proof. \( |P| = 2^n + 2^{3n/4}. \)

For $|P| \geq 16^k$, there is a $k$-element subset $X \subset P$ such that $X$ is either a $k$-cup or a $k$-cap, and the regions $T_1, \ldots, T_{k-1}$ from the support of $X$ satisfies $|T_i \cap P| \geq \frac{|P|}{2^{40k}}$.

Proof. $|P| = 2^{n+2n^{3/4}}$.

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**Proof.** \(|P| = 2^{n+2n^{3/4}}\).

For $|P| \geq 16^k$, there is a $k$-element subset $X \subset P$ such that $X$ is either a $k$-cup or a $k$-cap, and the regions $T_1, \ldots, T_{k-1}$ from the support of $X$ satisfies $|T_i \cap P| \geq \frac{|P|}{240k}$.

Proof. $|P| = 2^{n + 2n^{3/4}}$. 

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For $|P| \geq 16^k$, there is a $k$-element subset $X \subset P$ such that $X$ is either a $k$-cup or a $k$-cap, and the regions $T_1, \ldots, T_{k-1}$ from the support of $X$ satisfies $|T_i \cap P| \geq \frac{|P|}{2^{40k}}$.

Proof. $|P| = 2^{n+2n^{3/4}}$, $k = n^{2/3}$, $|T_i \cap P| \geq 2^{n+2n^{3/4} - 40n^{2/3}}$. 

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Proof. $|P| = 2^{n+2n^{3/4}}$, $k = n^{2/3}$, $|T_i \cap P| \geq 2^{n+2n^{3/4}-40n^{2/3}}$. 

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On the Erdős-Szekeres convex polygon problem
Define partial order of $P_i = T_i \cap P$, where $p \prec q$ iff $q \in \text{conv}(p \cup x_{i-1}x_{i+2})$
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\[ p \prec q \]
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On the Erdős-Szekeres convex polygon problem
\( N = 2^n + 2n^{3/4} \), \( k = n^{2/3} \). Set \( \alpha = n^{-1/4} \).

Dilworth’s Theorem: Each \( P_i \) contains either

1. Antichain size \( \left( \frac{N}{2^{40k}} \right)^\alpha = 2^{n^{3/4}} + 2^{n^{1/2}} - 40n^{5/12} \)

2. Chain size \( \left( \frac{N}{2^{40k}} \right)^{1-\alpha} = 2^n + n^{3/4} - 40n^{2/3} - 2n^{1/2} \)
Case 1: $\frac{\sqrt{k}}{2} = \frac{n^{1/3}}{2}$ of the $P_i$s are nonadjacent antichains.

Antichain size $\left( \frac{N}{240k} \right)^{\alpha} = 2^{n^{3/4}+2n^{1/2}-40n^{5/12}} \geq f(n, 2n^{2/3})$
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Antichain size \( \left( \frac{N}{2^{40k}} \right)^{\alpha} = 2^{n^{3/4}} + 2^{n^{1/2}} - 40n^{5/12} \geq f(n, 2n^{2/3}) \)

Union of all small caps is a cap (done).
Case 2: $\sqrt{k} = n^{1/3}$ of the $P_i$-s are consecutive chains.

Chain of size $(\frac{N}{2^{40k}})^{1-\alpha} = 2^n + n^{3/4} - 40n^{2/3} - 2n^{1/2}$
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Chain of size $\left(\frac{N}{240k}\right)^{1-\alpha} = 2n^{1/n} - 40n^{2/3} - 2n^{1/2}$
Many large mutually avoiding sets

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Many large mutually avoiding sets

$$\sqrt{k} = n^{1/3}$$ of the $P_i$-s are consecutive chains.

Chain of size $\left(\frac{N}{2^{40}k}\right)^{1-\alpha} = 2n+n^{3/4}-40n^{2/3}-2n^{1/2} \geq f(n, 2n^{2/3})$
Case 2: $\sqrt{k} = n^{1/3}$ chains of size $2^{n+n^{3/4}-40n^{2/3}-2n^{1/2}}$

$$f(in^{2/3}, n - in^{2/3} + n^{2/3}) = \left(\frac{n + n^{2/3} - 4}{in^{2/3} - 2}\right) + 1 \leq 2^{n+2n^{2/3}}$$
$ES_d(n)$ = smallest integer such that any set of $ES_d(n)$ points in $\mathbb{R}^d$ in general position contains $n$ members in convex position.

**Theorem (Károlyi 2001)**

$$ES_d(n) \leq ES_{d-1}(n - 1) + 1.$$ 

$$ES_d(n) \leq ES(n - d + 2) + d - 2 \leq 2^{n+o(n)}.$$ 

**Conjecture**

Füredi: $ES_3(n) = 2^{c\sqrt{n}}$.

$$ES_d(n) = 2^{c_d n^{1/(d-1)}}.$$
Higher dimensions

Conjecture

Füredi: $ES_3(n) = 2^{c\sqrt{n}}$.

(????) $ES_d(n) = 2^{c_d n^{1/(d-1)}}$.

Theorem (Károlyi-Valtr 2003)

$ES_d(n) \geq 2^{cn^{1/(d-1)}}$.

$ES_d(n) \leq 2^{n+o(n)}$. 

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Mutually avoiding sets in $\mathbb{R}^d$

**Theorem (Aronov-Erdős-Goddard-Kleitman-Klugerman-Pach-Schulman 1991)**

Let $P$ be an $N$-element point set in general position in $\mathbb{R}^d$. Then there are subsets $A, B \subset P$ such that $|A|, |B| \geq N \frac{1}{d^2 - d + 1}$ and $A$ and $B$ are mutually avoiding.
Mutually avoiding sets in $\mathbb{R}^d$

Theorem (Aronov-Erdős-Goddard-Kleitman-Klugerman-Pach-Schulman 1991)

Let $P$ be an $N$-element point set in general position in $\mathbb{R}^d$, $d \geq 3$. Then there are subsets $A, B \subset P$ such that $|A|, |B| \geq N^{d^2 - d + 1}$ and $A$ and $B$ are mutually avoiding.

Theorem (Valtr 1994)

There is an $N$-element point set $P$ in $\mathbb{R}^d$ in general position that does not contain a pair of mutually avoiding sets of size more than $cN^{1 - \frac{1}{d}}$. 
Back in the plane

**Theorem** (Aronov-Erdős-Goddard-Kleitman-Klugerman-Pach-Schulman 1991)

*Every complete $n$-vertex geometric graph contains $\sqrt{n}$ pairwise crossing edges.*

**Conjecture**

*Every complete $n$-vertex geometric graph contains $n^{1-\epsilon}$ pairwise crossing edges.*
Ramsey approach

Conjecture

Füredi: $ES_3(n) = 2c\sqrt{n}$.

(???) $ES_d(n) = 2^{c_d} n^{1/(d-1)}$.

$V = \{N \text{ points in } \mathbb{R}^d \text{ in general position}\}$

$E = \{(d + 2)\text{-tuples NOT in convex position}\}$.

Theorem (Motzkin 1963)

Any set of $d + 3$ vertices (points) in $H$ induces 0, 2, 4 hyperedges.

$r_k(k + 1, t; n) =$ smallest integer $N$ such that every $N$-vertex $k$-uniform hypergraph $H$ contains either $k + 1$ vertices with $t$ edges, or an independent set of size $n$.

$ES_d(n) \leq r_{d+2}(d + 3, 5; n)$. 
$r_k(k + 1, t; n) =$ smallest integer $N$ such that every $N$-vertex $k$-uniform hypergraph $H$ contains either $k + 1$ vertices with $t$ edges, or an independent set of size $n$.

**Conjecture (Erdős-Hajnal 1964)**

$$r_k(k + 1, 5; n) = \text{twr}_4(cn) = 2^{2^{2cn}}.$$  

$$r_k(k + 1, t; n) = \text{twr}_{t-1}(cn)$$
Ramsey approach

Not a good approach: $ES_d(n) \leq r_{d+2}(d + 3, 5; n)$.

Theorem (Mubayi-S. 2016)

For $k \geq t + 2$

$$r_k(k + 1, t; n) = \operatorname{twr}_{t-1}(n^{k-t+1+o(1)})$$

$r^*_k(n) = \text{smallest integer } N \text{ such that every } N\text{-vertex } k\text{-uniform hypergraph } H \text{ with the property that every } k + 1 \text{ vertices induces 0, 2, 4 edges, contains an independent set of size } n.$

$ES_d(n) \leq r^*_d(n) \leq r_{d+2}(d + 3, 5; n)$.

Not much is known about $r^*_k(n)$. 
Thank you!