## Final Exam, Math 310, Fall 2016

**Problem 1.** Find an invertible matrix $P$ and a matrix $C$ of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ such that $\begin{pmatrix} 5 & -5 \\ 1 & 1 \end{pmatrix} = PCP^{-1}$. Also find $P^{-1}$.

**Solution.** We get the characteristic equation $(5-\lambda)(1-\lambda)+5=0$. We get $\lambda = 3 \pm i$. Set $\lambda = 3-i$. Then $C = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$. To find the corresponding eigenvectors, we solve the augmented matrix

$$\begin{pmatrix} 2+i & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence we have $x_2 = free$, and $x_1 = 5x_2/(2+i) = x_2(2-i)$. Therefore, we have the eigenvector $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. This implies

$P = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$, and $P^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$.

**Problem 2.** Let $W = Span\{\begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}\}$. Find the closest vector in $W$ to $b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. What is the distance between $b$ and $W$?

**Solution.** Applying Gram-Schmidt, we have $u_1 = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 5 \\ -8/5 \\ 16/5 \end{pmatrix}$. Closest vector is $v = \frac{u_1 \cdot b}{u_1 \cdot u_1} u_1 + \frac{u_2 \cdot b}{u_2 \cdot u_2} u_2$.

**Problem 3.** Find the inverse, if it exists, of the matrix $\begin{pmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{pmatrix}$

**Solution.** After performing Gaussian elimination, the matrix is not invertible.

**Problem 4.** Let $W$ be the set of polynomials of degree at most 2, such that $p(17) = 0$. Prove or disprove that $W$ is a subspace.

**Solution**. Yes, $W$ is a vector space. 1) The zero polynomial satsifies $p(17) = 0$. Also, if $p, q$ in $W$, then $p(17) + q(17) = 0$. Finally, if $p$ is in $W$, then $\alpha p(17) = 0$.

**Problem 5**. Find the least squares solution of $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}$, $x = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$.

**Solution.** By normalizing, we get the augmented matrix $\begin{pmatrix} 3 & 3 & 6 \\ 3 & 11 & 14 \end{pmatrix}$. Our solution set is $x_1 = 1$ and $x_2 = 1$.

**Problem 6**. Compute $A^{100}$, where $A = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$.

**Solution**. We get the characteristic equation $(7-\lambda)(1-\lambda)+8=0$. This gives $\lambda_1 = 3$ and $\lambda_2 = 5$. For finding the eigenvector $v_1$ corresponding to $\lambda_1$, we find nontrivial solutions to the augmented matrix $\begin{pmatrix} 4 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We have $x_2 = free$ and $x_1 = -x_2/2$. This gives $v_1 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$.

For the other eigenvector $v_2$, we find nontrivial solutions to the augmented matrix $\begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. This gives $x_2 = free$ and $x_1 = -x_2$. Hence $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Therefore

$$A = \begin{pmatrix} -1/2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -2 & -1 \end{pmatrix}$$

This implies

$$A^{100} = \begin{pmatrix} -1/2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^{100} & 0 \\ 0 & 5^{100} \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -2 & -1 \end{pmatrix}$$

**Problem 7**. Over polynomials of degree at most 1, find the change-of-coordinate matrix from the basis $\mathcal{B} = \{1-2t, 3-5t\}$ to $\mathcal{C} = \{4, 2+t\}$. Find the $\mathcal{B}$-coordinates of $1+t$

**Solution.** Converting the polynomials into vectors, gives $B = \{\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \end{pmatrix}\}$, and $C = \{\begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\}$. Thus the change-of-coordinate matrix is $\begin{pmatrix} 5/4 & 13/4 \\ -2 & -5 \end{pmatrix}$. For the second part, we need to solve the augmented matrix

$$\begin{pmatrix} 1 & 3 & 1 \\ -2 & -5 & 1 \end{pmatrix}$$

This gives $x_1 = -8$ and $x_2 = 3$. Thus the $B$ coordinate of $1+t$ is $(-8, 3)$.

**Problem 8**. Suppose $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 9$. Compute $\begin{vmatrix} 2a-g & 2b-h & 2c-i \\ d+3g & e+3h & f+3i \\ 4g+a+d & 4h+b+e & 4i+c+f \end{vmatrix}$. Justify your answer.

**Solution**.

$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 3 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 2a-g & 2b-h & 2c-i \\ d+3g & e+3h & f+3i \\ 4g+a+d & 4h+b+e & 4i+c+f \end{pmatrix}.$$

Hence the determinant is $3 \cdot 9 = 27$

**Problem 9.** Let $c = \begin{pmatrix} 4/3 \\ -1 \\ 2/3 \end{pmatrix}$ and $d = \begin{pmatrix} 5 \\ 6 \\ -1 \end{pmatrix}$. Find the unit vector $u$ in the direction of $c$. Then show that $d$ is orthogonal to $c$.

**Solution.** $u = c/||c|| = \sqrt{9/29}\begin{pmatrix} 4/3 \\ -1 \\ 2/3 \end{pmatrix}$. $d \cdot c = 0$.