Problem 1. Recall the simplicial partition theorem: For any \( n \)-element point set \( P \subset \mathbb{R}^2 \) and a parameter \( r : 1 < r < n \), there is a partition \( P = P_1 \cup \cdots \cup P_t \), such that

1. \( P_i \subset \sigma_i \), where \( \sigma_i \) is a relatively open simplex,
2. \( t = O(r) \),
3. \( \frac{n}{r} \leq |P_i| \leq \frac{2n}{r} \) for all \( i \), and
4. any line \( \ell \) in the plane crosses at most \( O(\sqrt{r}) \) \( \sigma_i \)'s. Here, \( \ell \) crosses a \( \sigma_i \) if \( \ell \cap \sigma_i \neq \emptyset \), and \( \sigma_i \not\subset \ell \).

Use the simplicial partition theorem to prove the Szemerédi-Trotter theorem.

Problem 2. (a) Let \( P \) be an \( n \)-element point set in the plane and let \( k < \sqrt{n} \) be an integer parameter. Prove at most \( O(n^2/k) \) pairs of points of \( P \) lie on lines containing at least \( k \) points and at most \( \sqrt{n} \) points of \( P \).

(b) For \( K > \sqrt{n} \), prove that the number of pairs of points in \( P \) lying on lines with at least \( \sqrt{n} \) and at most \( K \) points is \( O(Kn) \).

(c) Prove that there is an absolute constant \( c > 0 \) such that for any \( n \)-element point set \( P \) in the plane, at least \( cn^2 \) distinct lines are determined by \( P \) or there is a line containing at least \( cn \) points of \( P \).

Problem 3. Let \( q \geq 2 \) be an integer and let \( K = mq + 2 \) for integer \( m \geq 1 \). Prove that every sufficiently large set \( P \subset \mathbb{R}^2 \) in general position contains a \( k \)-point convex subset \( Y \) such that the number of points of \( P \) in the interior of \( \text{conv}(Y) \) is divisible by \( q \). Hint: Use Ramsey’s theorem.

Problem 4. Let \( G \) be the family of graphs such that \( G \in \mathcal{G} \) if and only if \( G \) is the intersection of a family of axis-parallel rectangular frames in the plane. Prove that \( \mathcal{G} \) is not chi-bounded.

Problem 5. Prove using pigeonhole that the Ramsey number \( b(n,n) \leq 4^n n \). Partial credit will be given for proofs using Kovari-Sos-Turan.