Homework 1, Math 261C, Discrete Geometry Due May 11

Problem 1. Recall the simplicial partition theorem: For any n-element point set $P \subset \mathbb{R}^2$ and a parameter $r : 1 < r < n$, there is a partition $P = P_1 \cup \cdots \cup P_t$, such that

1. $P_i \subset \sigma_i$, where σ_i is a relatively open simplex,
2. $t = O(r)$,
3. $\frac{n}{r} \leq |P_i| \leq \frac{2n}{r}$ for all i, and
4. any line ℓ in the plane crosses at most $O(\sqrt{r})$ σ_i's. Here, ℓ *crosses* a σ_i if $\ell \cap \sigma_i \neq \emptyset$, and $\sigma_i \not\subset \ell$.

Use the simplicial partition theorem to prove the Szemered-Trotter theorem.

Problem 2. (a) Let P be an n-element point set in the plane and let $k < \sqrt{m}$ be an integer parameter. Prove at most $O(n^2/k)$ pairs of points of P lie on lines containing at least k points and at most $\sqrt{n}$ points of P.

(b) For $K > \sqrt{n}$, prove that the number of pairs of points in P lying on lines with at least $\sqrt{n}$ and at most K points is $O(Kn)$.

(c) Prove that there is an absolute constant $c > 0$ such that for any n-element point set P in the plane, at least cn^2 distinct lines are determined by P or there is a line containing at least cn points of P.

Problem 3. Let $q \geq 2$ be an integer and let $K = mq + 2$ for integer $m \geq 1$. Prove that every sufficiently large set $P \subset \mathbb{R}^2$ in general position contains a k-point convex subset Y such that the number of points of P in the interior of $conv(Y)$ is divisible by q. Hint: Use Ramsey's theorem.

Problem 4. Let $\mathcal{G}$ be the family of graphs such that $G \in \mathcal{G}$ if and only if G is the intersection of a family of axis-parallel rectangular frames in the plane. Prove that $\mathcal{G}$ is not chi-bounded.

Problem 5. Prove using pigeonhole that the Ramsey number $b(n, n) \leq 4^n n$. Partial credit will be given for proofs using Kovari-Sos-Turan.