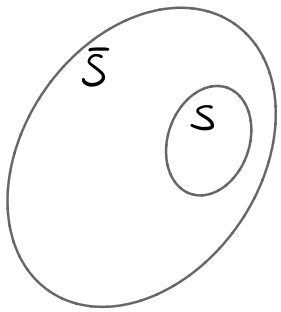
The Principle of Inclusion-Exclusion

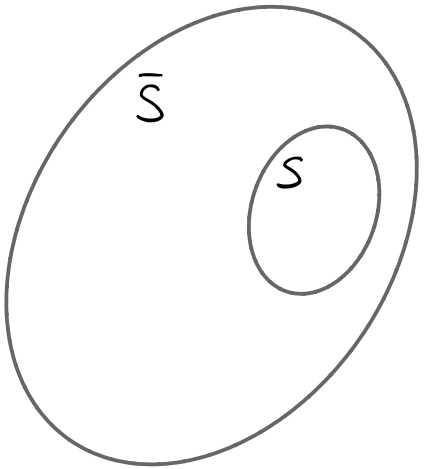
- Fundamental Problem in Combinatorics: given a (finite) set S, determine $|S|$.

- Different strategies, depending on how the set S is "given."

- If S is contained in a larger set $\overline{S}$, you can try "count the complement" strategy.



- At the level of sets,



$$\bar{S} = S \sqcup \bar{S} \setminus S.$$

- At the level of numbers,

$$|\bar{S}| = |S| + |\bar{S} \setminus S|.$$

- If you know how to compute $|\bar{S}|$ and $|\bar{S} \setminus S|$, you can solve for $|S|$ ("Subtraction Principle"):

$$|S| = |\bar{S}| - |\bar{S} \setminus S|.$$

Problem: How many n-bit strings contain at least one 1?

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Solution:

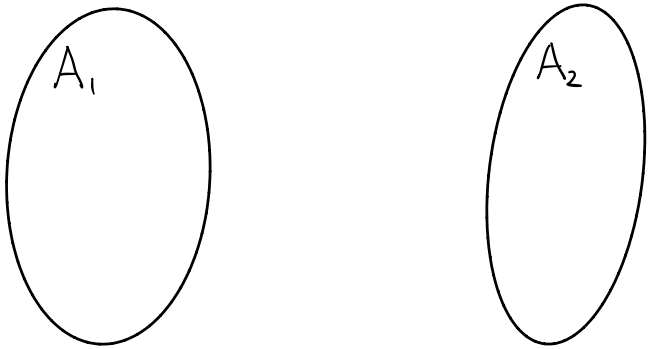
- $S = \{\text{n-bit strings with at least one 1}\}$.
- $\bar{S} = \{\text{n-bit strings}\}$. Clearly, $|\bar{S}| = 2^n$.
- $\bar{S} \setminus S = \{\text{n-bit strings with no 1}\} = \{000\ldots0\}$. Clearly, $|\bar{S} \setminus S| = 1$.
- Subtraction Principle:

$$|S| = |\bar{S}| - |\bar{S} \setminus S| = 2^n - 1.$$

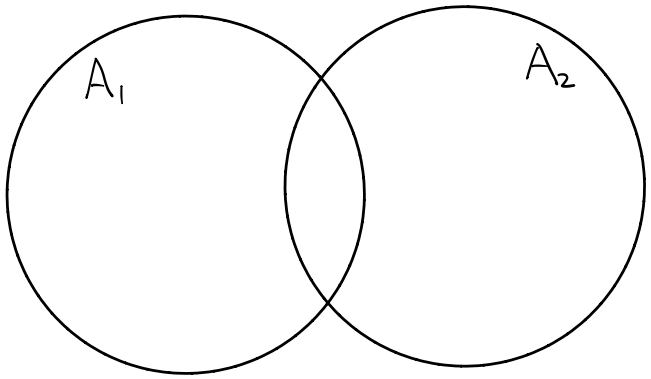
- Subtraction Principle really just an instance of "Partition Principle"

$$A = A_1 \sqcup A_2 \quad \Rightarrow \quad |A_1 \sqcup A_2| = |A_1| + |A_2|.$$

- Notation: $A_1 \sqcup A_2$ means the union of two disjoint sets, $A_1 \cap A_2 = \emptyset$,



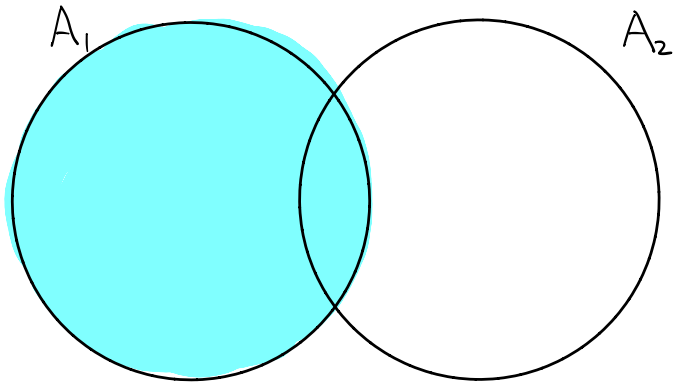
- What happens when $A = A_1 \cup A_2$ is the union of two non-disjoint sets?



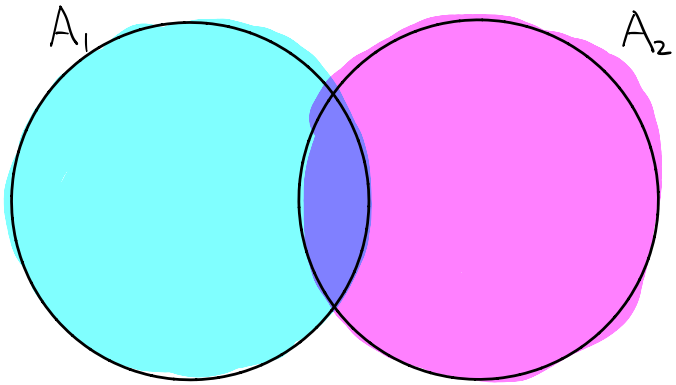
- What is the relationship between the numbers $|A_1 \cup A_2|$, $|A_1|$, $|A_2|$?

- Think of this as an "area" or "paving" computation.

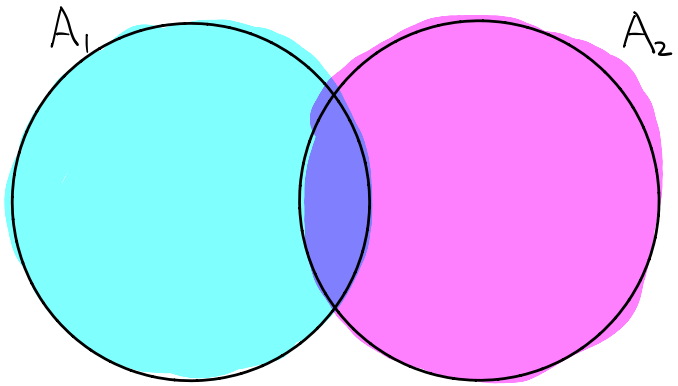
- First, pave oval A_1:



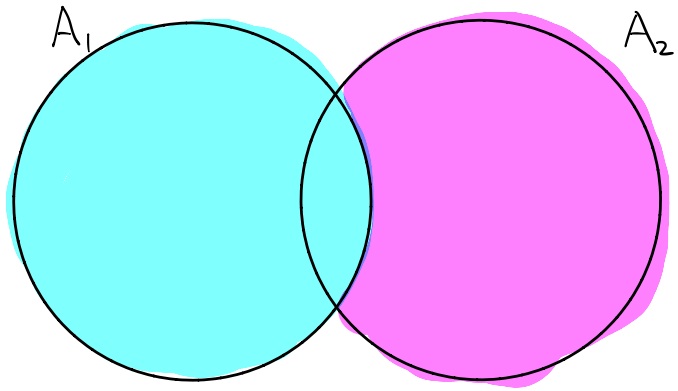
- Then pave oval A_2:



- The overlapping area was paved twice:



- To get a smooth paving, you have to strip one layer of pavement off the overlap:



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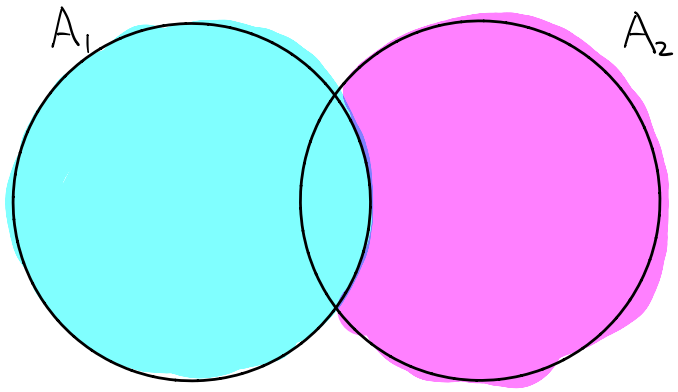

- In formulas,

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

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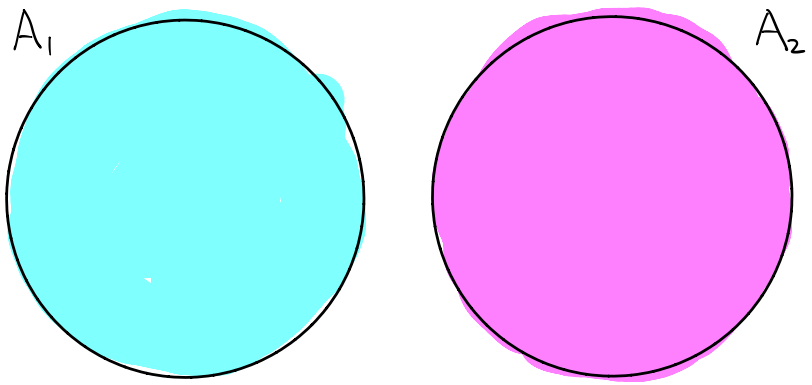
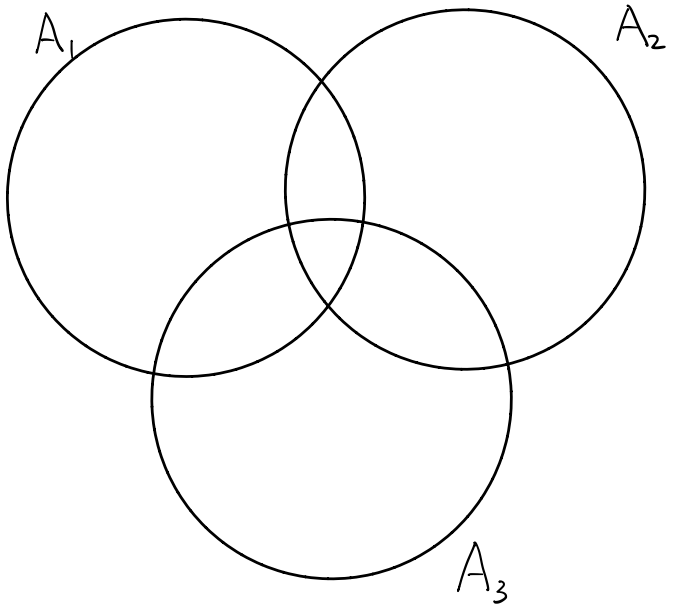
- Observe that this degenerates to the partitioning principle if $A_1 \cap A_2 = \emptyset$,



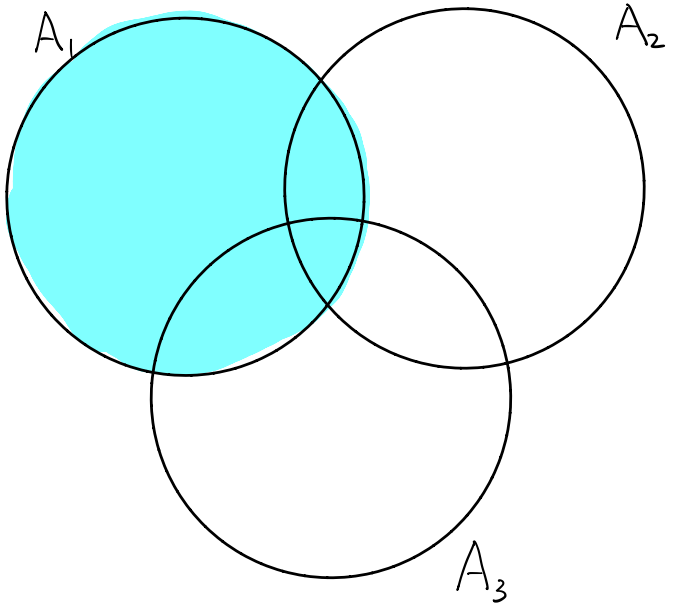
$$|A_1 \cup A_2| = |A_1| + |A_2| - 0.$$

- Consider now the case of three sets:



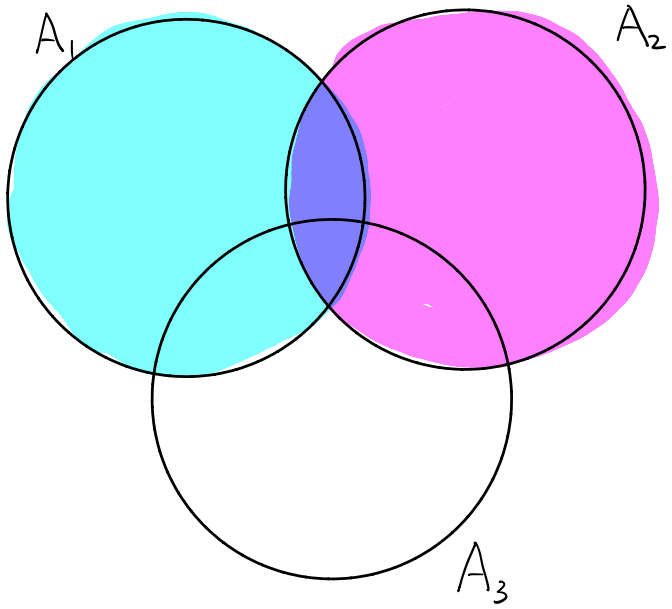
- We want to evenly pave the region delineated by these three circles.

- Start by paving A_1:



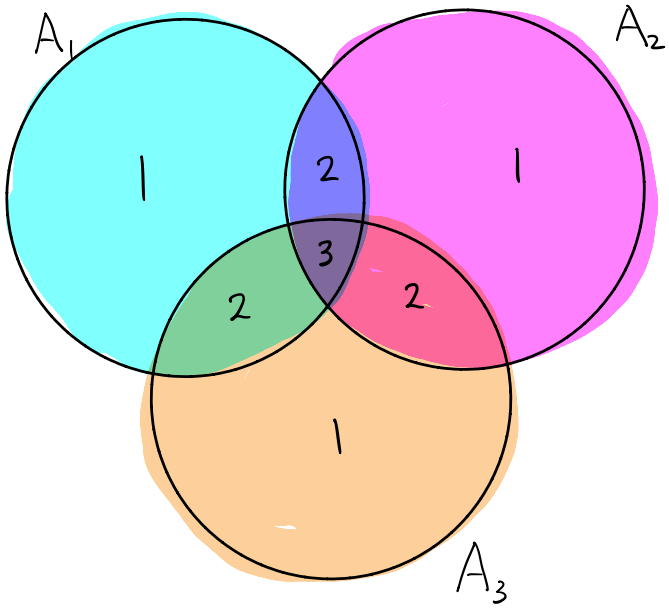
- No problem so far.

- Now pave A_2:



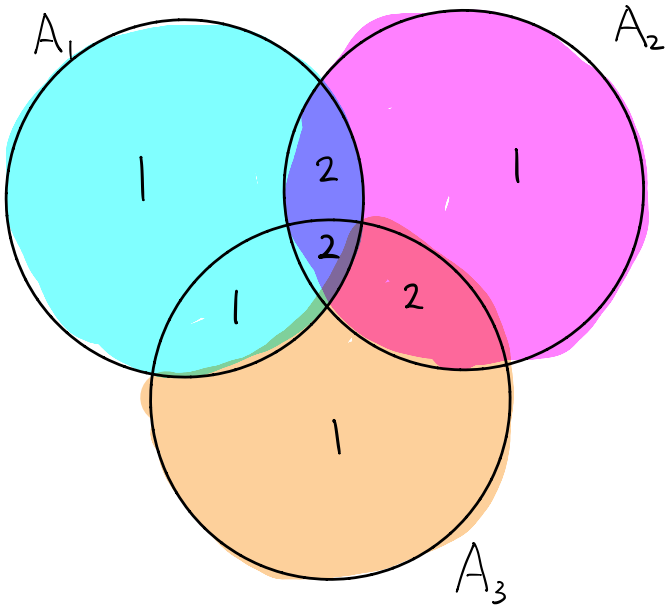
- Problem: the region $A_1 \cap A_2$ has been paved twice.

- Now pave A_3:

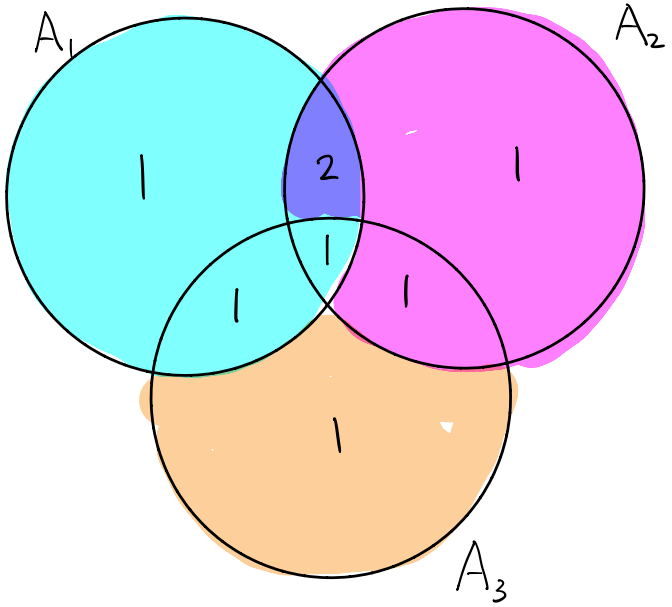


- Multiple issues:
  - The region $A_1 \cap A_2$ has been paved twice;
  - The region $A_1 \cap A_3$ has been paved twice;
  - The region $A_2 \cap A_3$ has been paved twice;
  - The region $A_1 \cap A_2 \cap A_3$ has been paved three times.

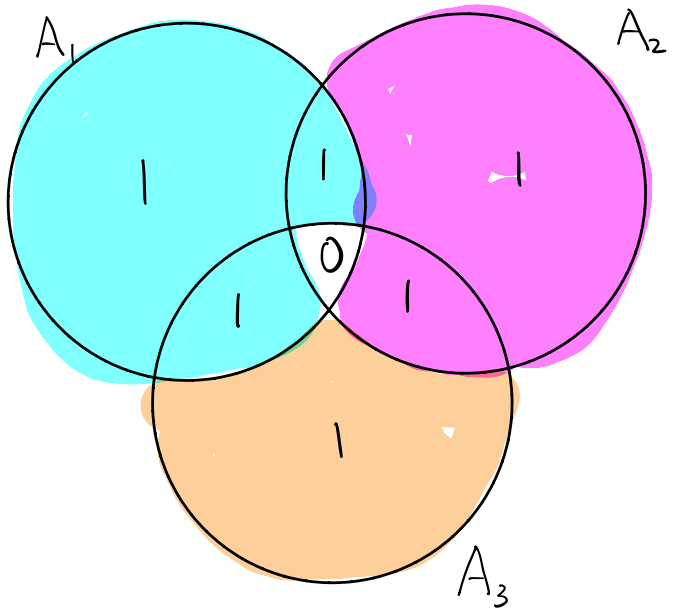
- Strip a layer of pavement off $A_1 \cap A_3$:



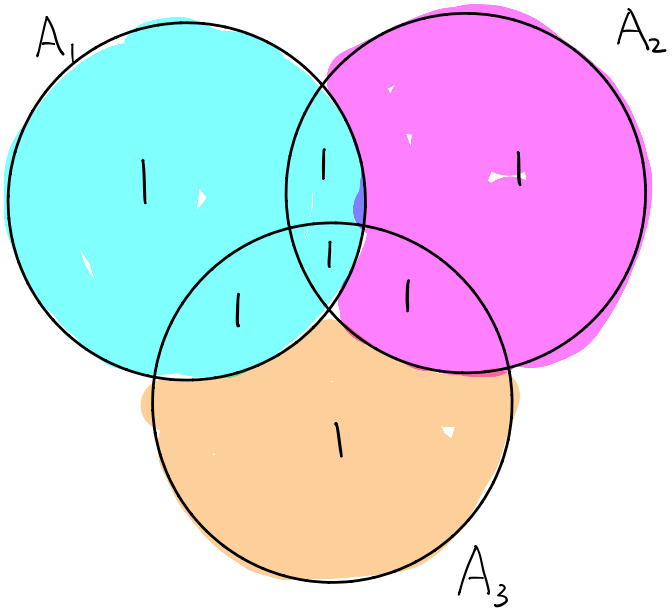
- Strip a layer of pavement off $A_2 \cap A_3$:



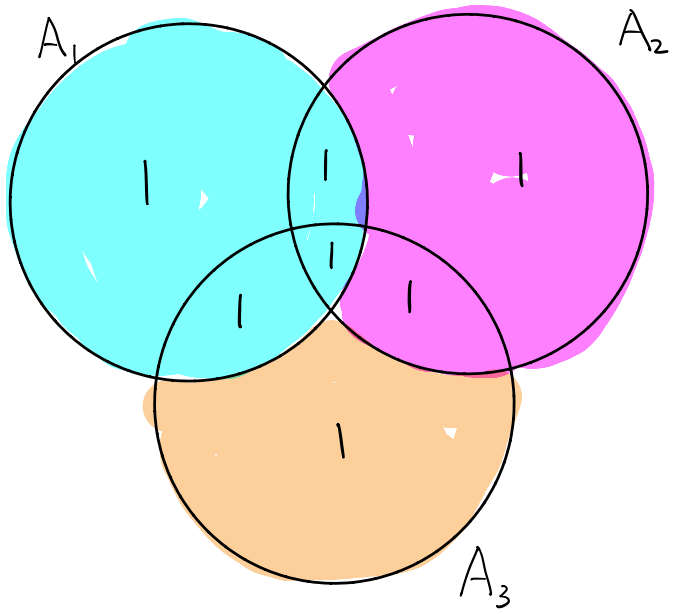
- Strip a layer of pavement off $A_1 \cap A_2$:



- Re-pave $A_1 \cap A_2 \cap A_3$,



- Re-pave $A_1 \cap A_2 \cap A_3$,



- Conclusion:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

- Extrapolation:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = ?$$

- Extrapolation:

$$|A_1 \cup A_2| = \sum \text{cardinality} - \sum \text{cardinality of intersection}$$

$$|A_1 \cup A_2 \cup A_3| = \sum \text{cardinality} - \sum \text{cardinality of intersection} + \sum \text{cardinality of triple intersection}$$

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = ?$$

- Extrapolation:

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = \sum \text{cardinality}$$

$$- \sum \text{cardinality of intersection}$$

$$+ \sum \text{cardinality of triple intersection}$$

$$- \sum \text{cardinality of quadruple intersection.}$$

Theorem (Principle of Inclusion-Exclusion)

For any positive integer n, and any finite sets $A_1, \ldots, A_n$,

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} (-1)^{i-1} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I| = i}} \left| \bigcap_{j \in I} A_j \right|.$$

Equivalently,

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{\substack{I \subset \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right|.$$

Example: $|A_1 \cup A_2 \cup A_3 \cup A_4|$ equals

$\{1\}$ $\{2\}$ $\{3\}$ $\{4\}$

$\{1,2\}$ $\{1,3\}$ $\{1,4\}$ $\{2,3\}$ $\{2,4\}$ $\{3,4\}$

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$$\underset{\{1\}}{+|A_1|} \quad \underset{\{2\}}{+|A_2|} \quad \underset{\{3\}}{+|A_3|} \quad \underset{\{4\}}{+|A_4|}$$

$$\underset{\{1,2\}}{-|A_1 \cap A_2|} \quad \underset{\{1,3\}}{-|A_1 \cap A_3|} \quad \underset{\{1,4\}}{-|A_1 \cap A_4|} \quad \underset{\{2,3\}}{-|A_2 \cap A_3|} \quad \underset{\{2,4\}}{-|A_2 \cap A_4|} \quad \underset{\{3,4\}}{-|A_3 \cap A_4|}$$

$$\underset{\{1,2,3\}}{+|A_1 \cap A_2 \cap A_3|} \quad \underset{\{1,2,4\}}{+|A_1 \cap A_2 \cap A_4|} \quad \underset{\{1,3,4\}}{+|A_1 \cap A_3 \cap A_4|} \quad \underset{\{2,3,4\}}{+|A_2 \cap A_3 \cap A_4|}$$

$$\underset{\{1,2,3,4\}}{-|A_1 \cap A_2 \cap A_3 \cap A_4|}$$

Proof:

- We want to prove that

$$|A_1 \cup \dots \cup A_n| = \sum_{i=1}^{n} (-1)^{i-1} \sum_{\substack{S \subseteq \{1,\dots,n\} \\ |S| = i}} \left| \bigcap_{j \in S} A_j \right|. \qquad (*)$$

- Choose an arbitrary element $x \in A_1 \cup \dots \cup A_n$.

- The LHS of $(*)$ counts x exactly once.

- How many times does the RHS of $(*)$ count x?

- Let r be the number of sets on the list $A_1, \ldots, A_n$ which contain x.

- Since $x \in A_1 \cup \ldots \cup A_n$, it must belong to at least one A_i, so $r \geq 1$. But r could be as large as n (if $x \in A_1, \ldots, x \in A_n$).

- Without loss in generality, suppose $x \in A_1, \ldots, x \in A_r$. For example, if r were 3 and $x \in A_1$, $x \in A_2$, $x \in A_n$, we could rename A_n "A_3" and rename A_3 "A_n."

- This means that, when considering how many times $(*)$ counts x, we need only consider the partial sum

$$\sum_{i=1}^{r} (-1)^{i-1} \sum_{\substack{S \subseteq \{1,\ldots,r\} \\ |S| = i}} \left| \bigcap_{j \in S} A_j \right|. \qquad (**)$$

- Consider the $i=1$ term in (**). The internal sum is

$$\sum_{\substack{S \subseteq \{1,\dots,r\} \\ |S|=1}} \left| \bigcap_{j \in S} A_j \right| = |A_1| + |A_2| + \dots + |A_r|,$$

which counts x r times.

- Consider the $i=2$ term in (**). The internal sum is

$$\begin{aligned} \sum_{\substack{S \subseteq \{1,\dots,r\} \\ |S|=2}} \left| \bigcap_{j \in S} A_j \right| = {} & |A_1 \cap A_2| + |A_1 \cap A_3| + \dots + |A_1 \cap A_r| \\ & + |A_2 \cap A_3| + \dots + |A_2 \cap A_r| \\ & \quad \ddots \qquad \vdots \\ & \qquad\qquad + |A_{r-1} \cap A_r|, \end{aligned}$$

which counts x $\binom{r}{2}$ times.

- In general, the i^{th} term of

$$\sum_{i=1}^{r} (-1)^{i-1} \sum_{\substack{S \subseteq \{1,\dots,r\} \\ |S| = i}} \left| \bigcap_{j \in S} A_j \right| \qquad (**)$$

counts x a total of $\binom{r}{i}$ times, once for each term of the internal sum, and with a sign of $(-1)^{i-1}$.

- Thus

$$|A_1 \cup \dots \cup A_n| = \sum_{i=1}^{n} (-1)^{i-1} \sum_{\substack{S \subseteq \{1,\dots,n\} \\ |S| = i}} \left| \bigcap_{j \in S} A_j \right| \qquad (*)$$

is true if and only if

$$1 = \sum_{i=1}^{r} (-1)^{i-1} \binom{r}{i} \qquad (***)$$

is true.

- Let us verify that (***) is correct.

- Clearly, (***) is equivalent to the identity

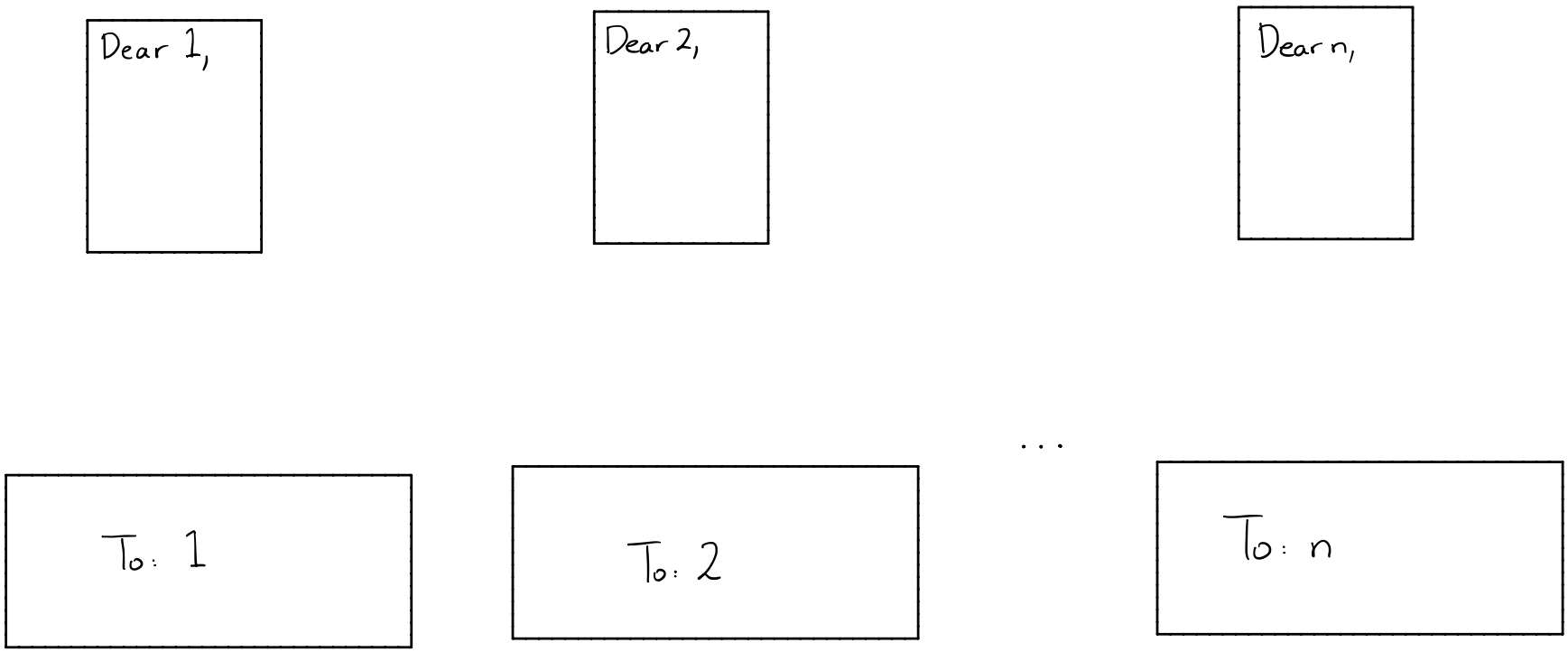
$$1 - \sum_{i=1}^{r} (-1)^{i-1} \binom{r}{i} = 0. \qquad (****)$$

- But the LHS of (****) is

$$1 - \sum_{i=1}^{r} (-1)^{i-1} \binom{r}{i} = 1 + \sum_{i=1}^{r} (-1)^{i} \binom{r}{i} = \sum_{i=0}^{r} (-1)^{i} \binom{r}{i} = \left((-1)+1\right)^{r} = 0.$$

– □

Problem: You are mailing letters to your friends $1, 2, \ldots, n$. You decide to close your eyes and randomly put the letters in the addressed envelopes. What is the probability that everyone gets the wrong letter?



Solution:

- Each way of placing the letters in the envelopes defines a permutation $\pi: \{1, \ldots, n\} \to \{1, \ldots, n\}$.

- The formula $\pi(i) = j$ means "letter i goes in envelope j."

- The probability we want to calculate is thus

$$p_n = \frac{|\{\pi \in S_n : \pi(i) \neq i \text{ for all } 1 \leq i \leq n\}|}{|S_n|},$$

where S_n is the set of all permutations $\pi: \{1, \ldots, n\} \to \{1, \ldots, n\}$.

- The denominator of p_n is $|S_n| = n!$; it remains to calculate the numerator.

- By the Subtraction Principle,

$$|\{\pi \in S_n : \pi(i) \neq i \text{ for all } 1 \leq i \leq n\}| = n! - |\{\pi \in S_n : \pi(i) = i \text{ for some } 1 \leq i \leq n\}|.$$

- So, we need to compute the number $|\{\pi \in S_n : \pi(i) = i \text{ for some } 1 \leq i \leq n\}|$ of permutations that have at least one fixed point.

- This number is $|A_1 \cup A_2 \cup \ldots \cup A_n|$, where A_i is the set of permutations which map i to itself:

$$A_i = \{\pi \in S_n : \pi(i) = i\}.$$

- By PIE,

$$|A_1 \cup \ldots \cup A_n| = \sum_{i=1}^{n} (-1)^{i-1} \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S| = i}} \left| \bigcap_{j \in S} A_j \right|$$

- Now, for any nonempty $S \subseteq \{1,\ldots,n\}$, we have

$$\bigcap_{j \in S} A_j = \{\pi \in S_n : \pi(j) = j \text{ for each } j \in S\}.$$

- The cardinality of this intersection is thus

$$\left| \bigcap_{j \in S} A_j \right| = (n - |S|)!.$$

- Returning to the PIE formula, we thus have

$$|A_1 \cup \dots \cup A_n| = \sum_{i=1}^{n} (-1)^{i-1} \sum_{\substack{S \subseteq \{1,\dots,n\} \\ |S| = i}} (n-i)!$$

$$= \sum_{i=1}^{n} (-1)^{i-1} (n-i)! \sum_{\substack{S \subseteq \{1,\dots,n\} \\ |S| = i}} 1$$

$$= \sum_{i=1}^{n} (-1)^{i-1} (n-i)! \binom{n}{i}$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \frac{n!}{i!} .$$

- Returning to the Subtraction Principle formula, we thus find that the number of permutations with no fixed point is

$$n! - |A_1 \cup \dots \cup A_n| = n! - \sum_{i=1}^{n} (-1)^{i-1} \frac{n!}{i!}$$

$$= n! \left(1 + \sum_{i=1}^{n} (-1)^i \frac{1}{i!} \right)$$

$$= n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$