Disjoint faces in drawings of the complete graph

Andrew Suk (UC San Diego)

June 12, 2023

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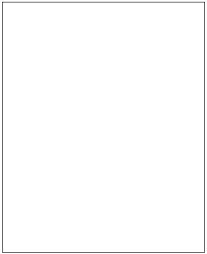

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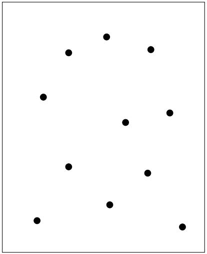

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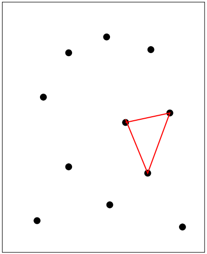

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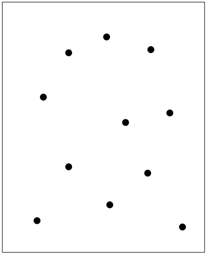

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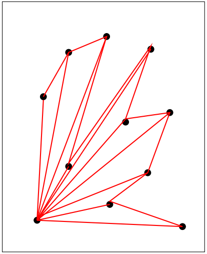

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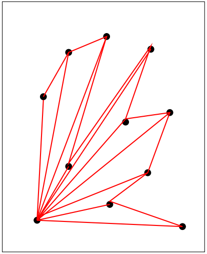

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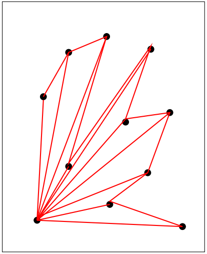

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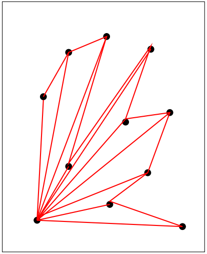

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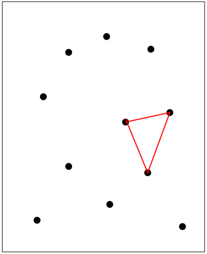

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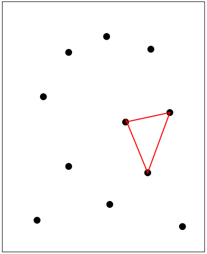

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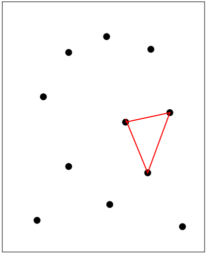

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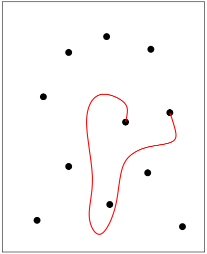

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Topological Graph $G = (V, E)$

V = points in the plane.

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Triangles and faces

k-**face:** Open bounded region of a non-self-intersecting k-cycle in K_n.

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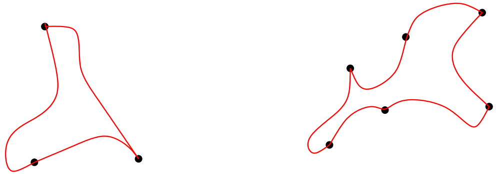

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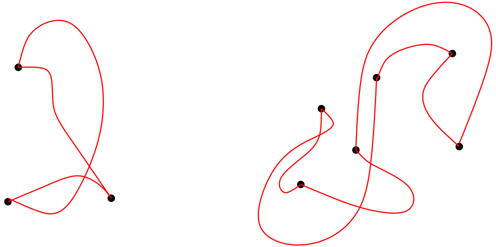

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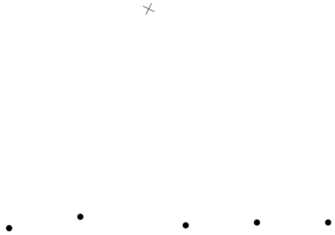

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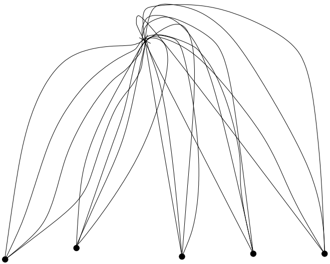

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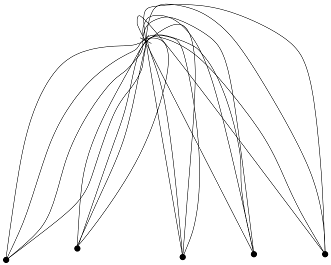

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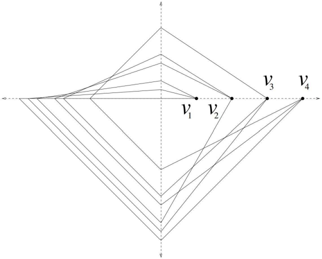

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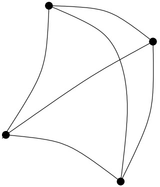
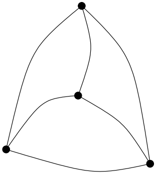
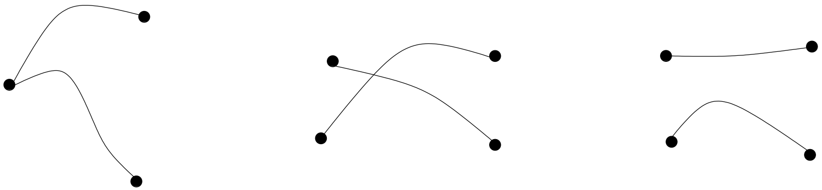




We will only consider simple topological graphs.



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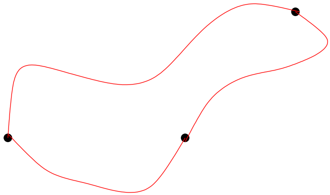

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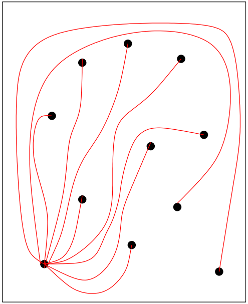

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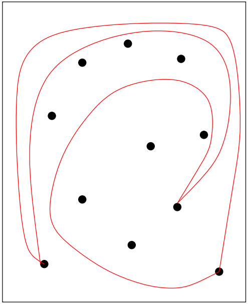

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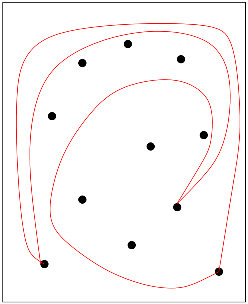

Answer: $\tilde{h}(n) \geq 1 - o(1)$.

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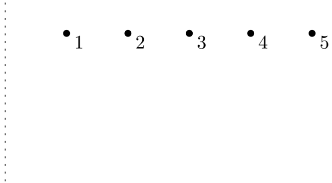

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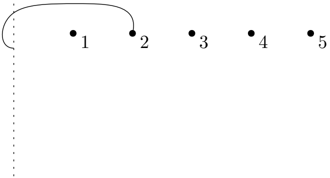

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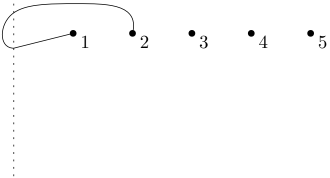

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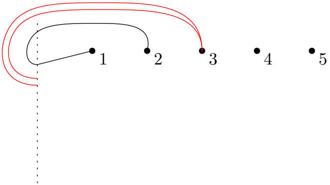

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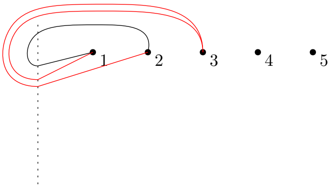

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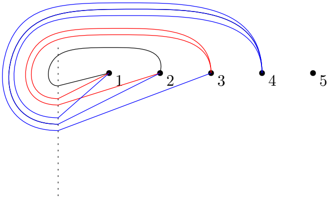

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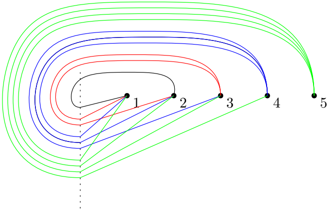

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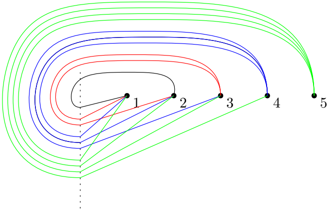

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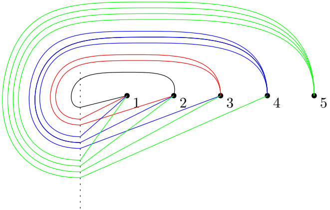

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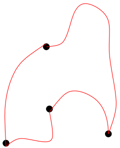
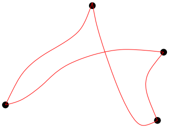
Question: What about the even faces? 4-faces?

Topological variant of Heilbronn's problem

Problem (Heilbronn)

What is the smallest $\tilde{h}_4(n)$ such that any **simple** *complete topological graph on n vertices in the unit square spans a* **4-face** *whose area is at most $\tilde{h}_4(n)$?*





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$$\tilde{h}_4(n) \leq O\left(\frac{1}{n^{1/3}}\right).$$

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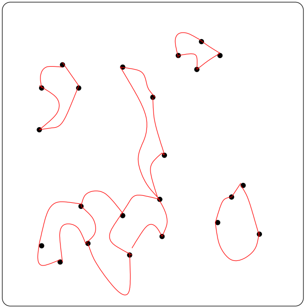
$K_n =$ 

Disjoint 4-faces: Ideas of the proof

Lemma (Ruiz-Vargas 2015)

There are two edges emanating out of v to the boundary, such that the edges lie completely inside the face.

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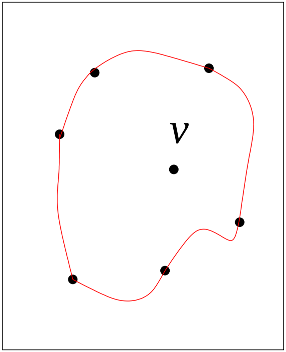

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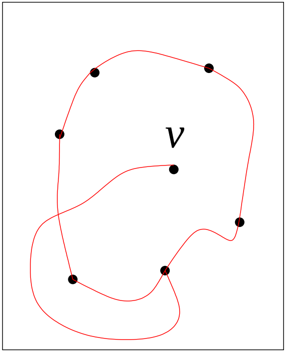

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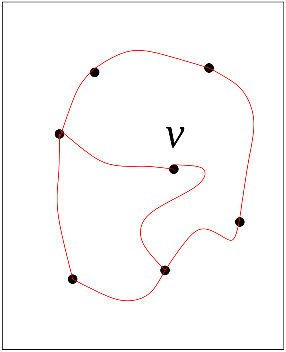

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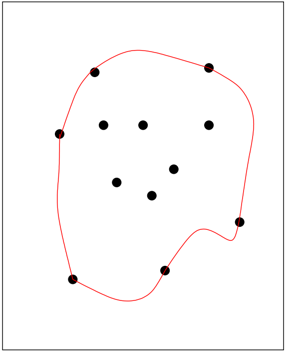
$K_n =$ 

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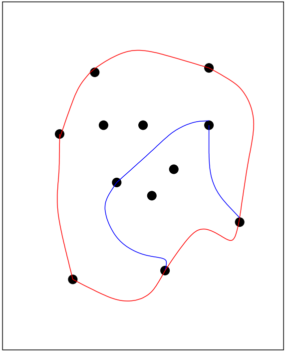

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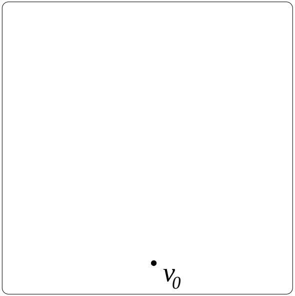

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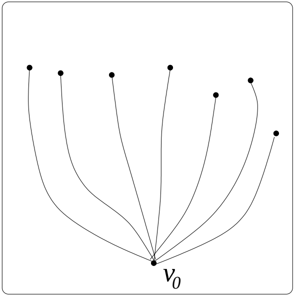

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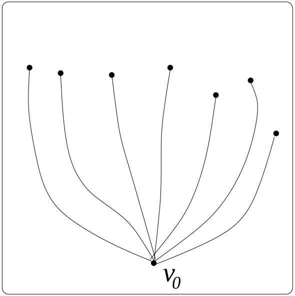

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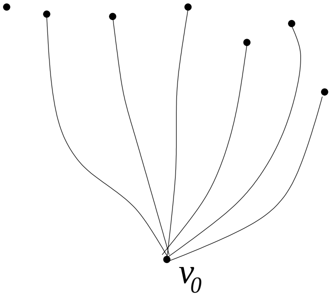

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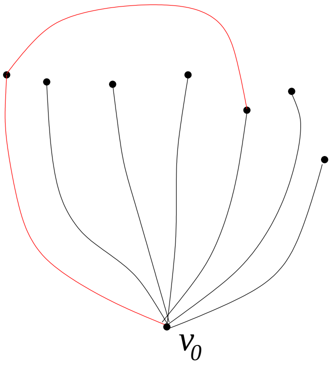

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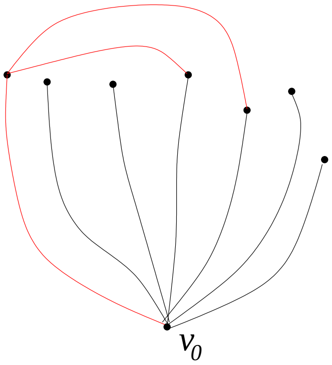

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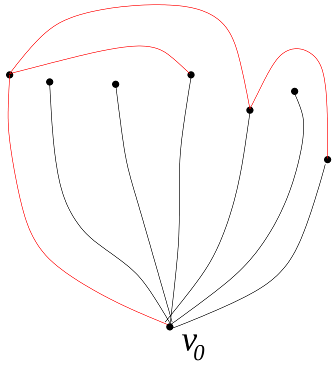

Sketch proof

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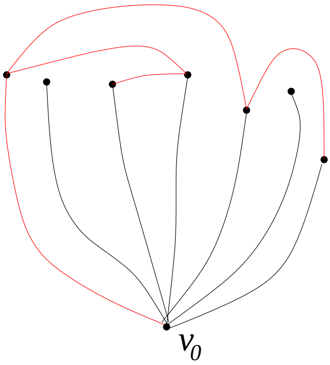

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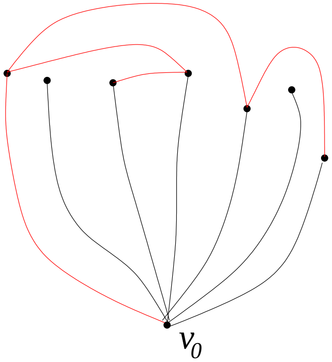
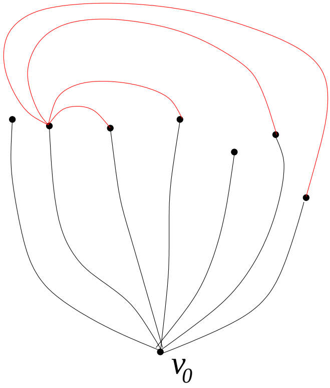


Planar graph with $\Theta(n)$ edges.

Sketch proof

Case 1. There is a vertex of degree $n^{1/3}$.



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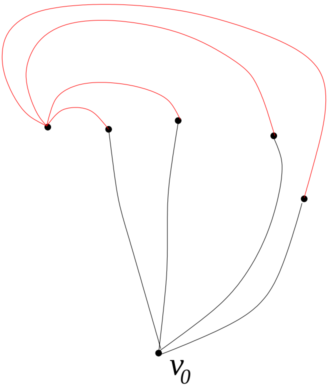
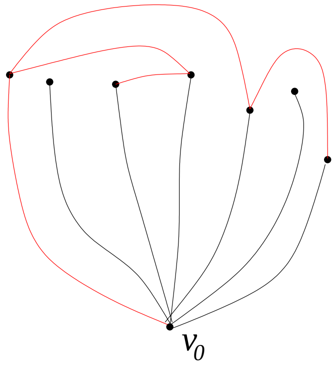


Planar $K_{2,cn^{1/3}}$ gives $\Theta(n^{1/3})$ pariwise disjoint 4-faces.

Sketch proof

Case 2. No vertex has degree $n^{1/3}$, matching of size $\Theta(n^{2/3})$.



Sketch proof

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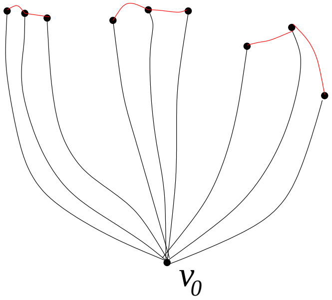

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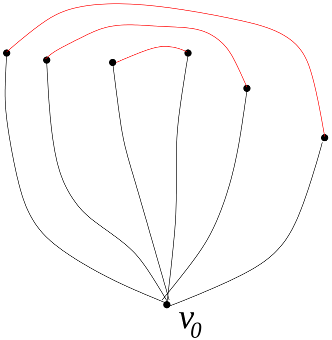

$\Theta(n^{1/3})$ nested sequence.

Sketch proof

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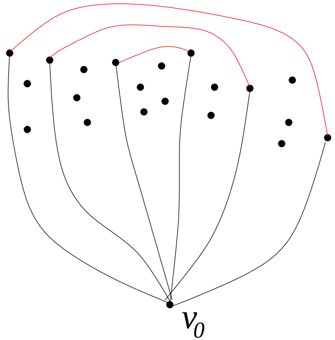

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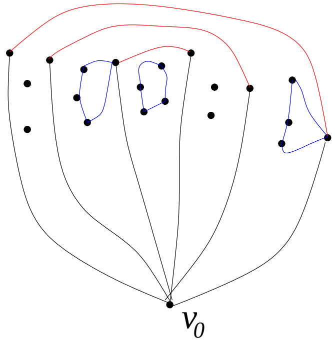

$\Theta(n^{1/3})$ nested sequence.

Open problems

$$\Omega\left(\frac{\log^{1/2} n}{n^{3/2}}\right) \leq \tilde{h}_4(n) \leq O\left(\frac{1}{n^{1/3}}\right)$$

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Problem

Can we improve the lower bound for complete simple topological graphs?

Open problems

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Problem

Can we improve the lower bound for complete simple topological graphs?

Problem

Does every complete simple topological graph on n vertices contain $\Omega(n)$ pairwise disjoint 4-faces?

Thank you!