Cliques and sunflowers under bounded VC-dimension

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Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$.

Definition

A set $S \subset V$ is **shattered** by $\mathcal{F}$ if for all $X \subset S$, there is an $A \in \mathcal{F}$ such that $S \cap A = X$.

Definition

The **VC-dimension of** $\mathcal{F}$ is the size of the largest subset $S \subset V$ that is shattered by $\mathcal{F}$.

VC-dimension of a graph

$G = (V, E)$, let $\mathcal{F} \subset 2^V$ such that $\mathcal{F} = \{N(v) : v \in V\}$.

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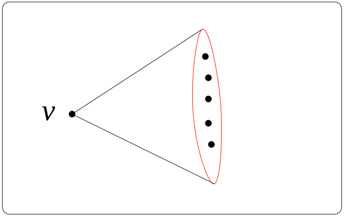

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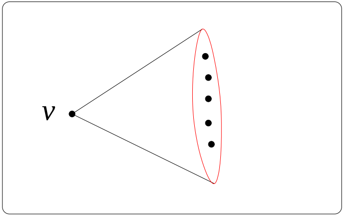
$G =$ 

Definition

The VC-dimension of G is the VC-dimension of $\mathcal{F}$.

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Problem

Can we substantially improve some of the classical theorems in extremal graph theory for graphs with bounded VC-dimension?

Classical theorems

1. **Ramsey's Theorem.** Every graph on n vertices contains a clique or independent set of size $c \log n$.

2. **Turán's Theorem.** Every $K_{2,2}$-free graph on n vertices has at most $cn^{3/2}$ edges.

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Problem

Can we improve these classical results for graphs with bounded VC-dimension?

Semi-algebraic vs VC-dimension

An application of the Milnor-Thom theorem:

Theorem

There are at most $2^{cn\log n}$ semi-algebraic graphs on n vertices and with complexity at most d, where $c = c(d)$.

Theorem (Anthony, Brightwell, Cooper 1995)

There are at least $2^{n^{2-\varepsilon}}$ graphs with VC-dimension at most d on n vertices, where $\varepsilon = \varepsilon(d)$.

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Classical theorems

2. **Turán's Theorem.** Every $K_{2,2}$-free graph on n vertices has at most $cn^{3/2}$ edges.
   - **Semi-algebraic graphs.** Improve to $O(n^{3/2-\varepsilon})$.
   - **Bounded VC-dimension.** No improvement. There are $K_{2,2}$-free graphs on n vertices with $\Omega(n^{3/2})$ edges.

First main result

In joint work with Jacob Fox and János Pach

- We establish tight bounds for multicolor Ramsey numbers for graphs with bounded VC-dimension.

Multicolor Ramsey numbers

Definition

For $m \geq 2$, The multicolor Ramsey number

$$r(\underbrace{3,\ldots,3}_{m \text{ times}})$$

is the minimum integer N such that for any m-coloring of the edges of K_N contains a monochromatic copy of K_3.

Known results

$$r(3,3) = 6 \qquad r(3,3,3) = 17 \qquad 51 \leq r(3,3,3,3) \leq 62$$

$$162 \leq r(3,3,3,3,3) \leq 307$$

$$2^m < r(\underbrace{3,\ldots,3}_{m \text{ times}}) < m!$$

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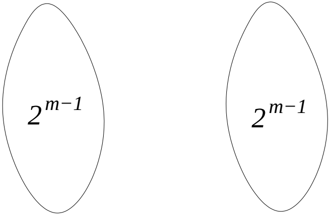

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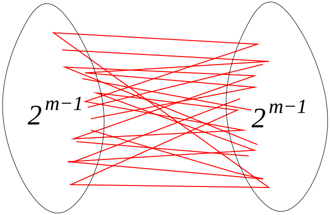

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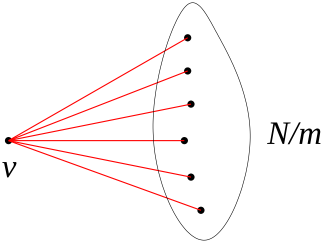

Best known bounds

Lower bound: Fredricksen-Sweet, Abbot-Moser.

Upper bound: Schur.

$$(3.199)^m < r(\underbrace{3,\ldots,3}_{m \text{ times}}) < 2^{O(m\log m)}$$

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Theorem (Fox-Pach-S., 2020)

The conjecture of true for semi-algebraic colorings with bounded complexity.

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Theorem (Fox-Pach-S., 2021)

The conjecture holds is true for colorings with bounded VC-dimension.

Color all edges of K_N with m colors.

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Bounded VC-dimension setting

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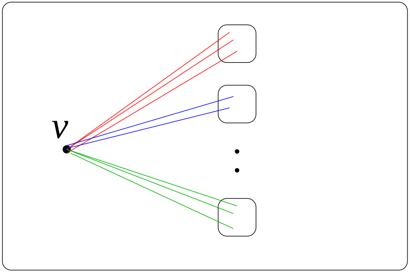
$K_N =$ 

Notation: $N_i(v) = \{u \in V : \chi(uv) = i\}$.

First main result

If we insist that the *m*-coloring has bounded VC-dimension:

$$\mathcal{F} = \{N_i(v) : v \in V, i \in [m]\}$$

$\mathcal{F}$ has VC-dimension at most $d = O(1)$.

Theorem (Fox-Pach-S. 2021)

For $m \geq 2$,

$$r_d(\underbrace{3, \ldots, 3}_{m \text{ times}}) = 2^{\Theta(m)}.$$

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Sketch of the proof:

$$r_d(\underbrace{3,\ldots,3}_{m \text{ times}}) \leq 2^{cm}, \qquad c = c(d)$$

Idea: We will use induction on m. Set $N = 2^{cm}$ and let V be an N-element vertex set.

$\chi : \binom{V}{2} \to \{1,2,\ldots,m\}$ and $\mathcal{F} = \{N_i(v) : v \in V, i \in [m]\}$.

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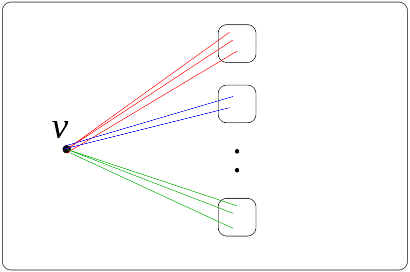
$K_N =$ 

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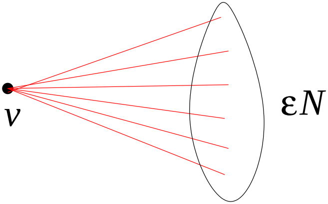

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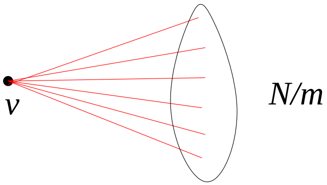
Not true: We can only assume $|N_i(v)| \geq N/m$ by pigeonhole.



$\mathcal{F} = \{N_i(v) : v \in V, i \in [m]\}$.

Crossing: Let $A \in \mathcal{F}$ and $u, v \in V$. Then A crosses $\{u, v\}$ if it contains one but not the other.

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Crossing pairs of vertices

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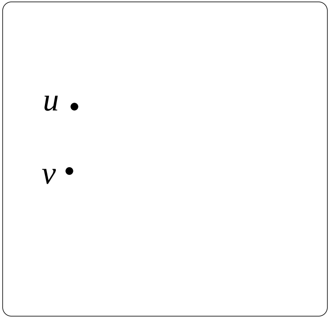

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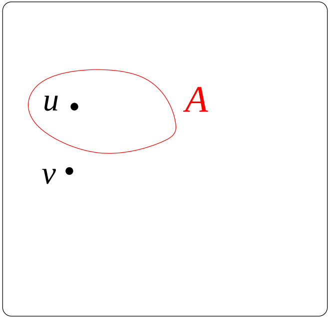

Partition Lemma

$\mathcal{F} = \{N_i(v) : v \in V, i \in [m]\}$, dual VC-dimension d.

Lemma

For any δ satisfying $1 \leq \delta \leq |\mathcal{F}|$, there is an equipartition $V = S_1 \cup \cdots \cup S_r$ of V into $r \leq c(|\mathcal{F}|/\delta)^d$ parts, such that any pair of vertices from the same part S_t is crossed by at most 2δ members of $\mathcal{F}$.

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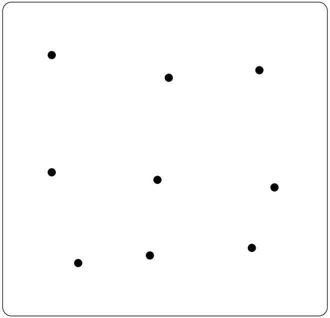

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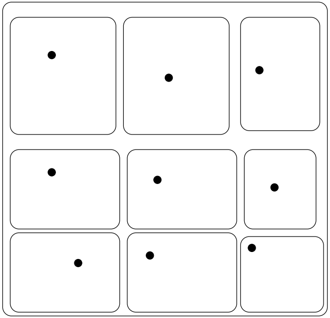
$K_N =$ 

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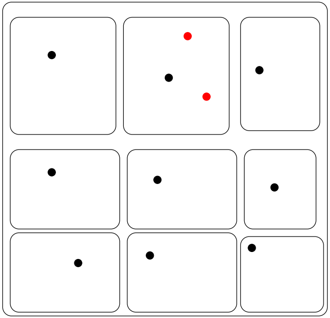
$K_N =$ 

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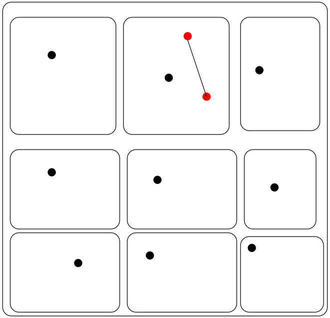
$K_N =$ 

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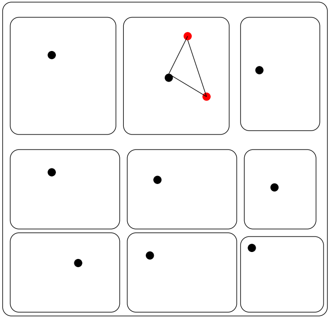

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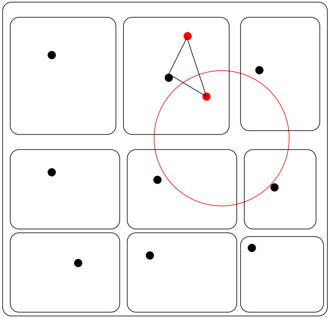
$K_N =$ 

Key observation:

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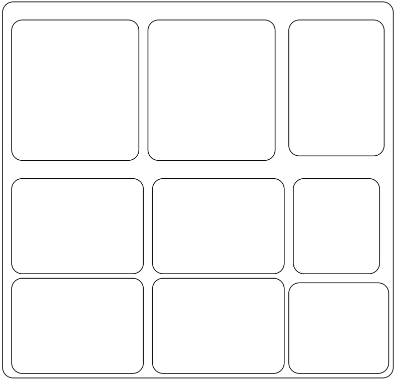

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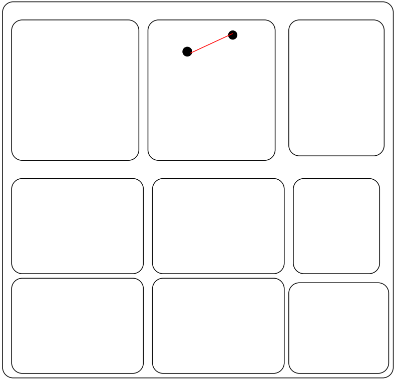
$K_N =$ 

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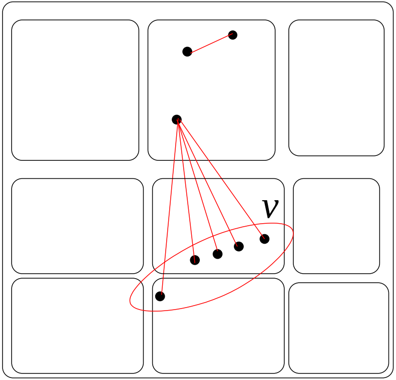
$K_N =$ 

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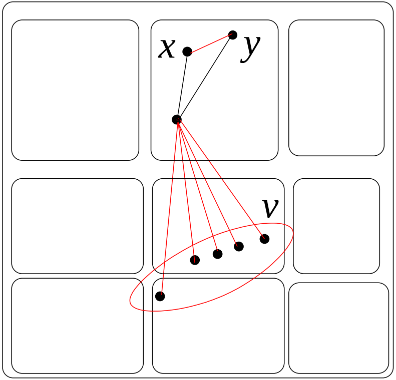
$K_N =$ 

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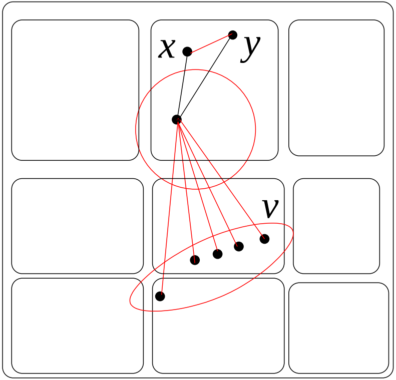
$K_N =$ 

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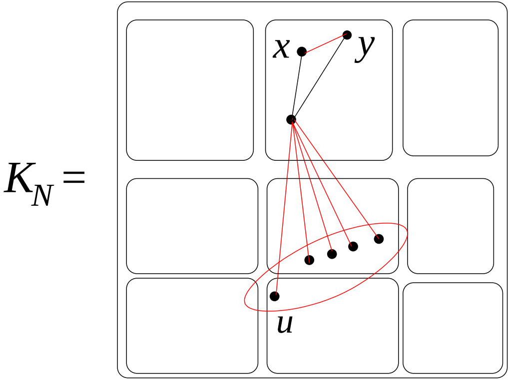

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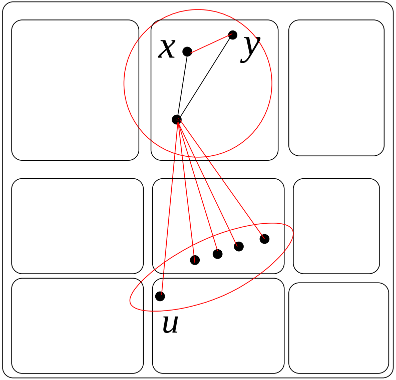

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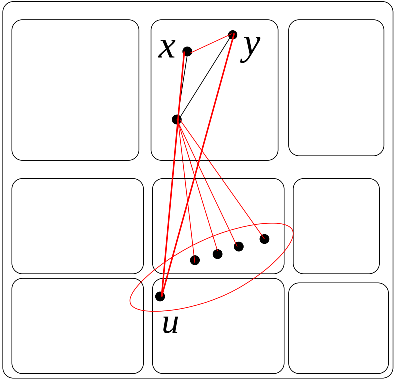

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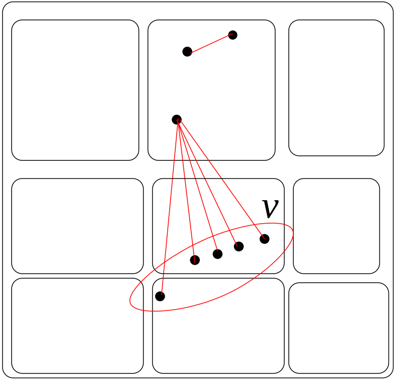
$K_N =$ 

Case 1. If a part is missing many colors, we are done by induction.

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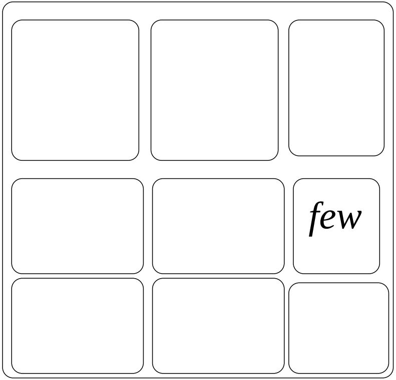

Partition Lemma

Case 2. Each part has many distinct colors

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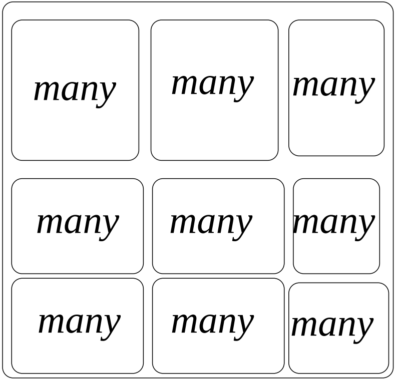

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Key observation: Otherwise, many vertices have small neighborhoods with respect to some colors.

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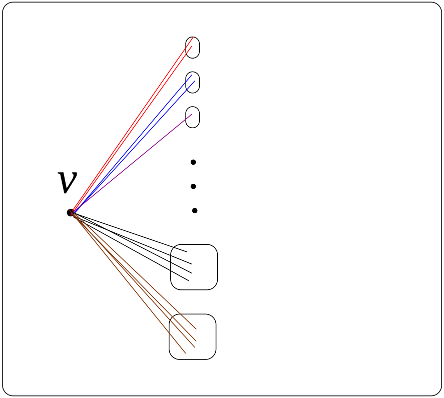

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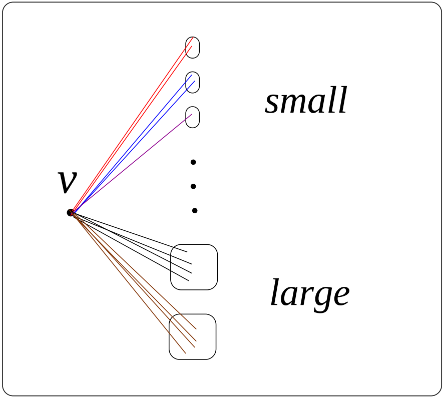

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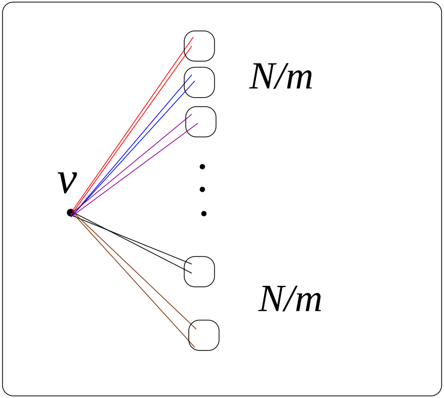

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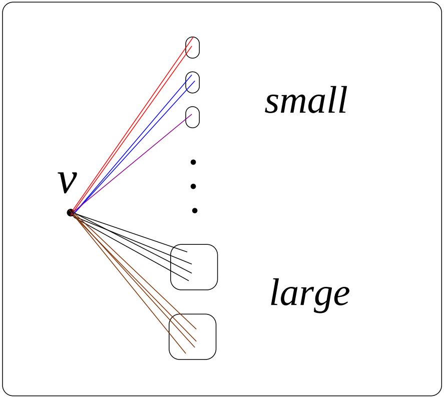

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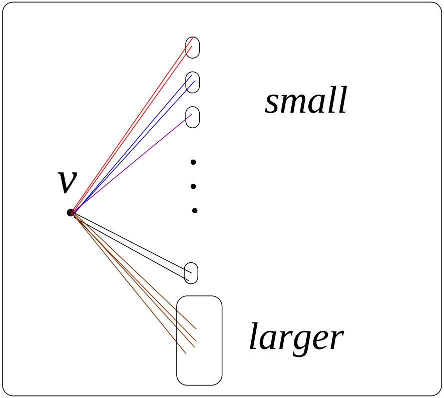
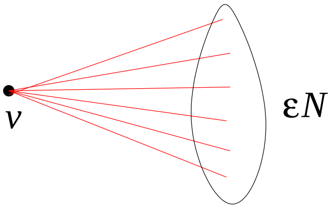


Partition Lemma

Goal: Find a vertex with large degree with respect to one color class.



Theorem (Fox-Pach-S. 2021)

For $d = O(1)$,

$$r_d(\underbrace{3, \ldots, 3}_{m \text{ times}}) = 2^{\Theta(m)}.$$

Theorem (Fredricksen-Sweet and Abbot-Moser, Schur)

$$(3.199)^m < r(\underbrace{3, \ldots, 3}_{m \text{ times}}) < 2^{O(m \log m)}.$$

Question. For what other classes of graphs can we improve the $2^{O(m \log m)}$ upper bound?

Improvement: Intersection size of sets

$\mathcal{F} \subset 2^X$, m-uniform.

1. **Vertices:** $V = \mathcal{F}$.
2. **Edge coloring:** For $A, B \in \mathcal{F}$, color (A, B) with color $i \in \{0, 1, \ldots, m-1\}$ if $|A \cap B| = i$.

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1. Schur: $2^{cm \log m}$.
2. Alweiss-Lovett-Wu-Zhang: $2^{cm \log \log m}$.

Sunflowers

V = ground set

$\mathcal{F} \subset \binom{V}{m}$, m-uniform.

$A_1, \ldots, A_p \in \mathcal{F}$ for a p-**sunflower** if $A_i \cap A_j = A_k \cap A_\ell$

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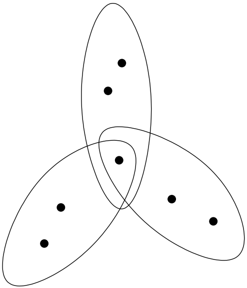

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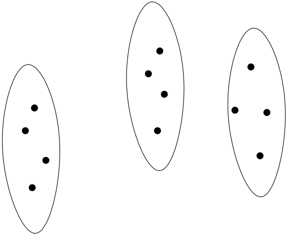

Erdős-Rado sunflower conjecture

Theorem (Erdős-Rado)

Let $\mathcal{F} \subset \binom{V}{m}$ that does not contain a 3-sunflower. Then

$$|\mathcal{F}| \leq m!2^m = 2^{O(m \log m)}.$$

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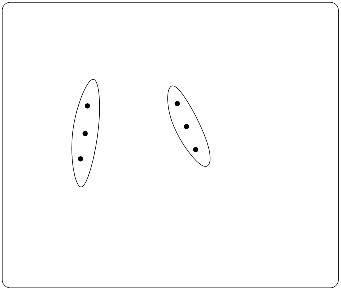
$\mathcal{F} =$ 

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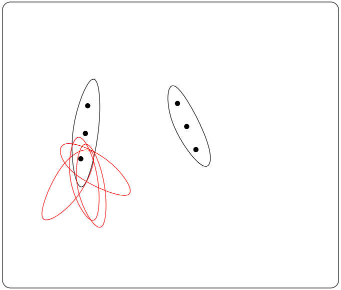
$\mathcal{F} =$ 

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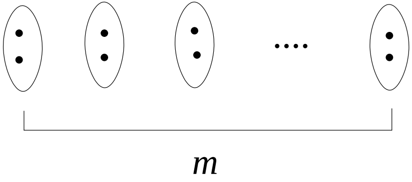

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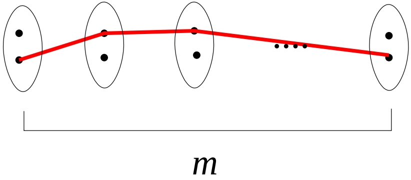

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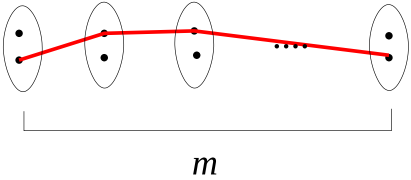

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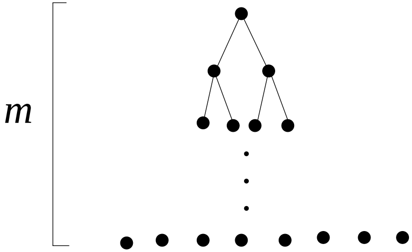
$\mathcal{F} = 2^{m-1}$, VC-dimension 1, no 3-sunflower.



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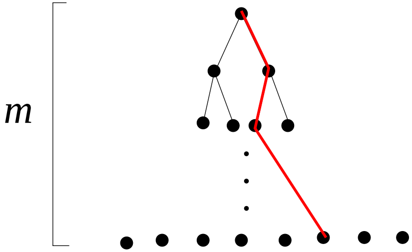

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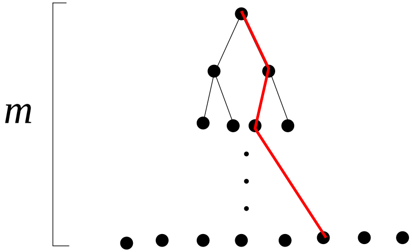

Can be realized geometrically: $V =$ points in the plane,

$\mathcal{F} =$ disks with m points inside.

Second main result

Theorem (Fox-Pach-S. 2021)

Let $\mathcal{F} \subset \binom{V}{m}$, such that $\mathcal{F}$ has VC-dimension $d = O(1)$ and no 3-sunflower. Then

$$|\mathcal{F}| \leq 2^{O(m(2d)^{2\log^* m})}.$$

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Skecth of Proof. Induction on m.

Let $\mathcal{F} \subset \binom{V}{m}$ with VC-dimension at most d and no 3-sunflower.

$$f_d(m) = 2^{cm(2d)^{2\log^* m}}.$$

$$|\mathcal{F}| \leq f_d(m).$$

$\mathcal{F} \subset \binom{V}{m}$, VC-dimension d, no 3-sunflower, $|\mathcal{F}| > f_d(m)$.

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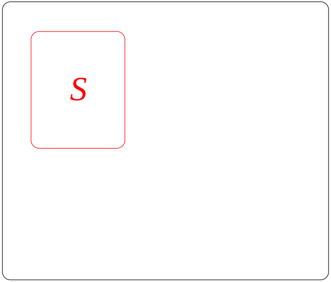

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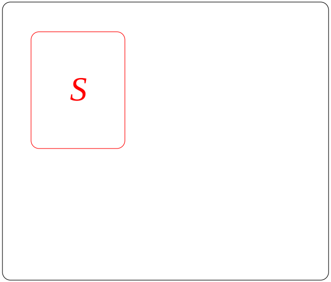
$\mathcal{F} =$ 

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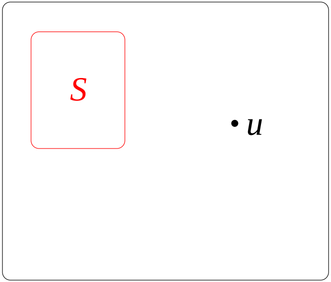
$\mathcal{F} =$ 

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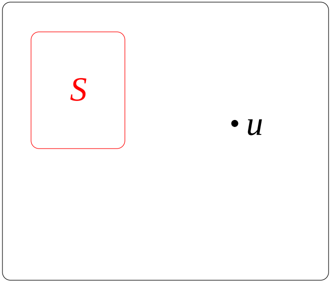

$$sd(u) \leq \sum_{v \in S} d(v) \leq m|\mathcal{F}| \qquad d(u) \leq (m/s)|\mathcal{F}|$$

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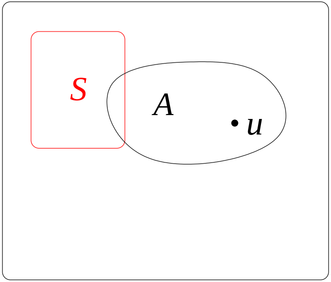

A intersects at most $(m^2/s)|\mathcal{F}|$ outside of S.

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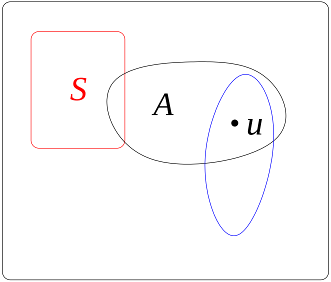

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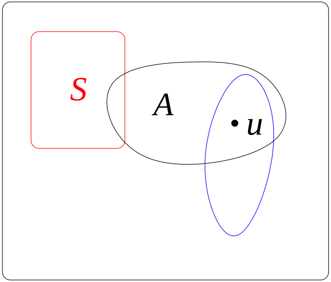
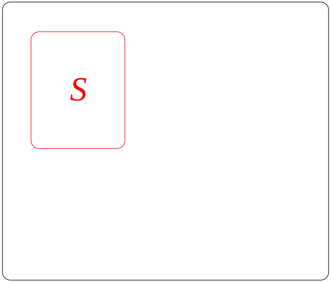
$\mathcal{F} =$ 

At least $(1 - \frac{6m^2}{s})\binom{|\mathcal{F}|}{3}$ triples are pairwise disjoint outside of S.

Case 1: For at least $|\mathcal{F}|/2$ sets $A \in \mathcal{F}$, $|A \cap S| \leq \log m$.

$\mathcal{F} =$ 

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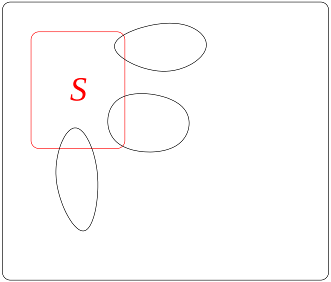
$\mathcal{F} =$ 

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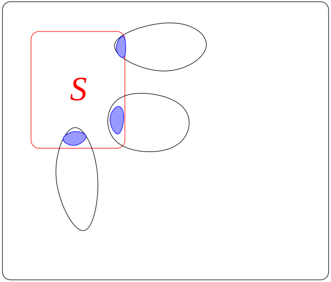

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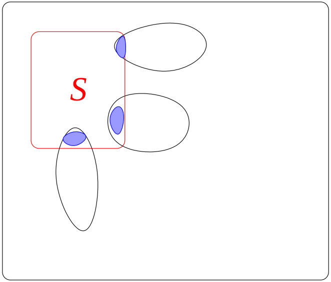
$\mathcal{F} =$ 

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1. $\mathcal{F}'$ is a multiset system
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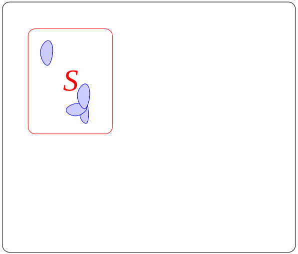
$\mathcal{F} =$ 

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There are at least

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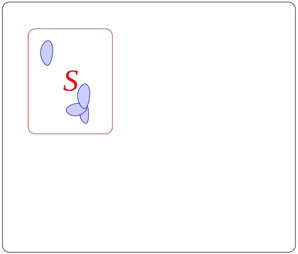
$\mathcal{F} =$ 

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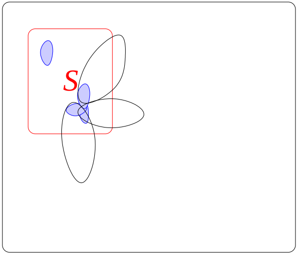
$\mathcal{F} =$ 

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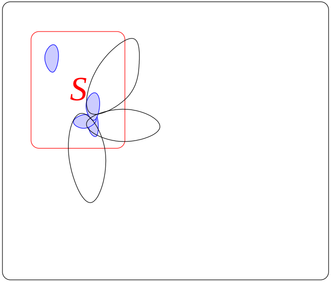
$\mathcal{F} =$ 

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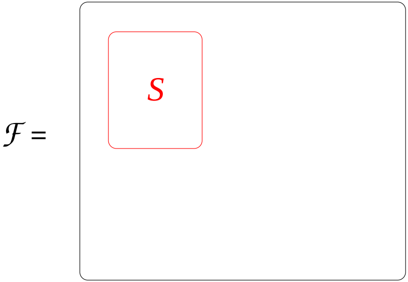

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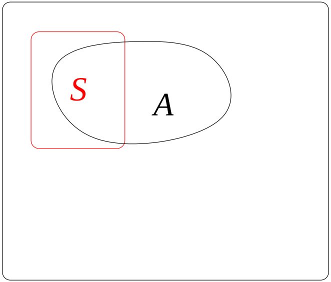
$\mathcal{F} =$ 

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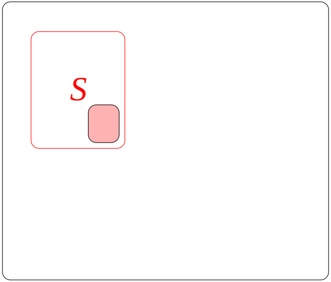
$\mathcal{F} =$ 

Sauer-Shelah: At most s^d distinct intersections with S.

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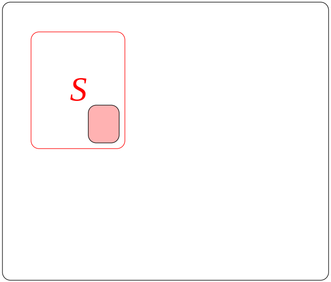

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$\exists S' \subset S$, $|S'| > \log m$.

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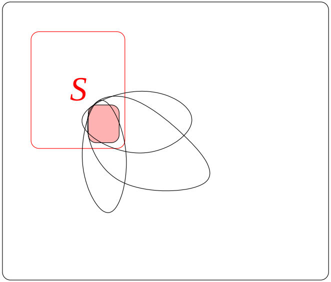
$\mathcal{F} =$ 

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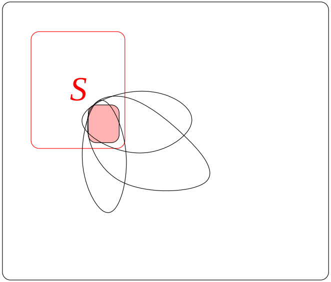
$\mathcal{F} =$ 

$$\frac{|\mathcal{F}|}{2s^d} \leq f_d(m - \log m) \leq 2^{c(m - \log m)(2d)^{2\log^* m}}$$

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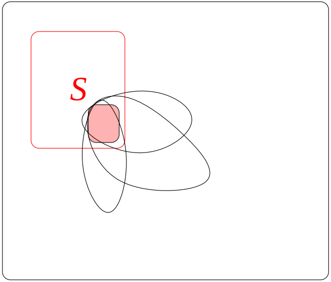
$\mathcal{F} =$ 

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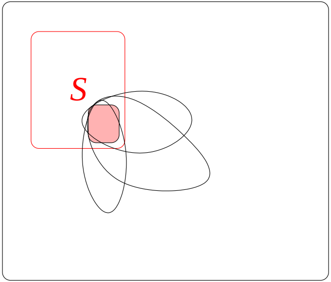
$\mathcal{F} =$ 

$$f_d(m) < |\mathcal{F}| \leq 2^{cm(2d)^{2\log^* m}} = f_d(m).$$

□

Open problems

Theorem (Fox-Pach-S. 2021)

Let $\mathcal{F} \subset \binom{V}{m}$, such that $\mathcal{F}$ has VC-dimension $d = O(1)$ and no 3-sunflower. Then

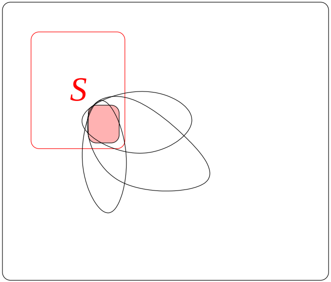
$$|\mathcal{F}| \leq 2^{O(m(2d)^{2\log^* m})}.$$

Questions

1. Semi-algebraic setting? I.e., points in spheres in $\mathbb{R}^d$.
2. (Weak delta-system) What about 3 sets that pairwise intersect with the same size?
3. Multicolor Ramsey numbers: What if each color class has bounded VC-dimension?

Thank you!

$$s = 100m^2(f_d(\log m))^2 = (100m^2)2^{2c\log m(2d)^{2(\log^* m-1)}}.$$

$\mathcal{F} =$ 

$$|\mathcal{F}| \leq 2s^d 2^{cm(2d)^{2\log^* m} - c\log m(2d)^{2\log^* m}}$$

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