Ramsey results for graphs with bounded VC-dimension

Andrew Suk (UC San Diego)

February 6, 2020

Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

Definition

A set $S \subset V$ is **shattered** by $\mathcal{F}$ if for all $X \subset S$, there is an $A \in \mathcal{F}$ such that $S \cap A = X$.

$\mathcal{F}$ =

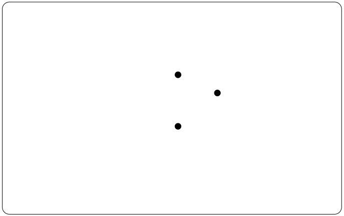
Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

Definition

A set $S \subset V$ is **shattered** by $\mathcal{F}$ if for all $X \subset S$, there is an $A \in \mathcal{F}$ such that $S \cap A = X$.

$\mathcal{F} =$

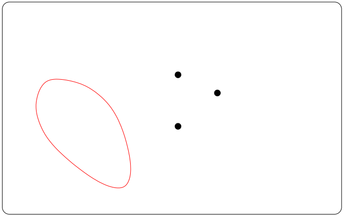


Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

Definition

A set $S \subset V$ is **shattered** by $\mathcal{F}$ if for all $X \subset S$, there is an $A \in \mathcal{F}$ such that $S \cap A = X$.

$\mathcal{F} =$ 

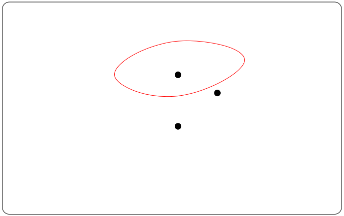
Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

Definition

A set $S \subset V$ is **shattered** by $\mathcal{F}$ if for all $X \subset S$, there is an $A \in \mathcal{F}$ such that $S \cap A = X$.

$\mathcal{F} =$



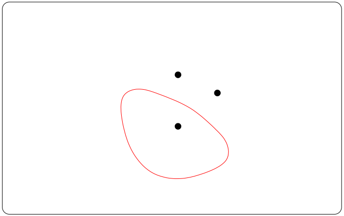
Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

Definition

A set $S \subset V$ is **shattered** by $\mathcal{F}$ if for all $X \subset S$, there is an $A \in \mathcal{F}$ such that $S \cap A = X$.

$\mathcal{F} =$

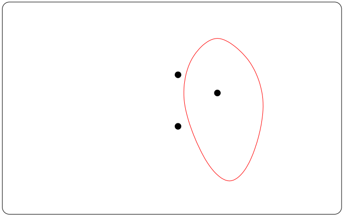


Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

Definition

A set $S \subset V$ is **shattered** by $\mathcal{F}$ if for all $X \subset S$, there is an $A \in \mathcal{F}$ such that $S \cap A = X$.

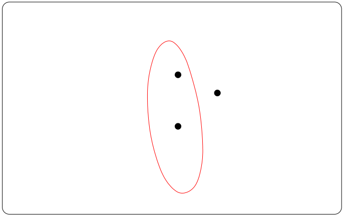
$\mathcal{F} =$ 

Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

Definition

A set $S \subset V$ is **shattered** by $\mathcal{F}$ if for all $X \subset S$, there is an $A \in \mathcal{F}$ such that $S \cap A = X$.

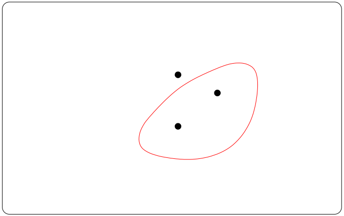
$\mathcal{F} =$ 

Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

Definition

A set $S \subset V$ is **shattered** by $\mathcal{F}$ if for all $X \subset S$, there is an $A \in \mathcal{F}$ such that $S \cap A = X$.

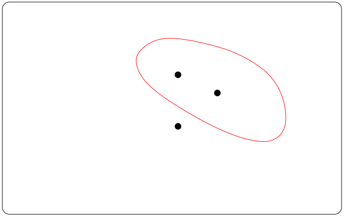
$\mathcal{F} =$ 

Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

Definition

A set $S \subset V$ is **shattered** by $\mathcal{F}$ if for all $X \subset S$, there is an $A \in \mathcal{F}$ such that $S \cap A = X$.

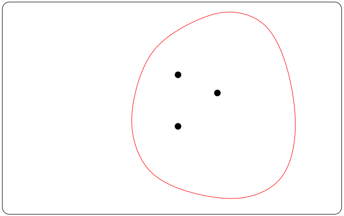
$\mathcal{F} =$ 

Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

Definition

A set $S \subset V$ is **shattered** by $\mathcal{F}$ if for all $X \subset S$, there is an $A \in \mathcal{F}$ such that $S \cap A = X$.

$\mathcal{F} =$ 

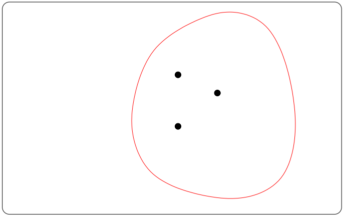
Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

Definition

The **VC-dimension of** $\mathcal{F}$ is the size of the largest subset $S \subset V$ that is shattered by $\mathcal{F}$.

$\mathcal{F} =$



VC-dimension of a graph

$G = (V, E)$, let $\mathcal{F} \subset 2^V$ such that $\mathcal{F} = \{N(v) : v \in V\}$.
$|V| = |\mathcal{F}| = n$

$G =$

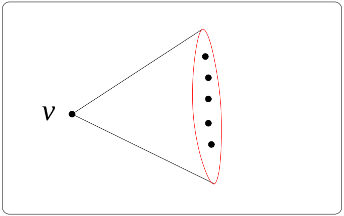
VC-dimension of a graph

$G = (V, E)$, let $\mathcal{F} \subset 2^V$ such that $\mathcal{F} = \{N(v) : v \in V\}$.
$|V| = |\mathcal{F}| = n$

G =

VC-dimension of a graph

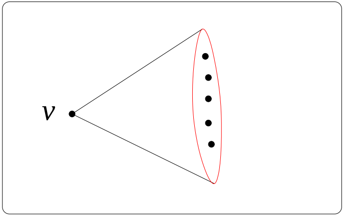
$G = (V, E)$, let $\mathcal{F} \subset 2^V$ such that $\mathcal{F} = \{N(v) : v \in V\}$.
$|V| = |\mathcal{F}| = n$

G = 

VC-dimension of a graph

$G = (V, E)$, let $\mathcal{F} \subset 2^V$ such that $\mathcal{F} = \{N(v) : v \in V\}$.
$|V| = |\mathcal{F}| = n$

G =



Definition

The VC-dimension of G is the VC-dimension of $\mathcal{F}$.

Examples of graphs with bounded VC-dimension

Examples of graphs with bounded VC-dimension

1. Intersection graphs of segments in the plane.

Examples of graphs with bounded VC-dimension

1. Intersection graphs of segments in the plane.
2. Unit distance graph of points in $\mathbb{R}^d$.

Examples of graphs with bounded VC-dimension

1. Intersection graphs of segments in the plane.

2. Unit distance graph of points in $\mathbb{R}^d$.

3. **Semi-algebraic graphs** with bounded complexity.
   - $V =$ points in $\mathbb{R}^d$
   - $E =$ defined by bounded degree polynomial inequalities.

Examples of graphs with bounded VC-dimension

1. Intersection graphs of segments in the plane.
2. Unit distance graph of points in $\mathbb{R}^d$.
3. **Semi-algebraic graphs** with bounded complexity.
   - $V =$ points in $\mathbb{R}^d$
   - $E =$ defined by bounded degree polynomial inequalities.
4. Intersection graph of pseudo-segments in the plane.

Examples of graphs with bounded VC-dimension

1. Intersection graphs of segments in the plane.
2. Unit distance graph of points in $\mathbb{R}^d$.
3. **Semi-algebraic graphs** with bounded complexity.
   - $V =$ points in $\mathbb{R}^d$
   - $E =$ defined by bounded degree polynomial inequalities.
4. Intersection graph of pseudo-segments in the plane.

Problem

Can we substantially improve some of the classical theorems in extremal graph theory for graphs with bounded VC-dimension?

Classical theorems

1. **Ramsey's Theorem.** Every graph on n vertices contains a clique or independent set of size $c \log n$.

2. **Turán's Theorem.** Every $K_{2,2}$-free graph on n vertices has at most $cn^{3/2}$ edges.

3. **Szemerédi's regularity lemma.**

Classical theorems

1. **Ramsey's Theorem.** Every graph on n vertices contains a clique or independent set of size $c \log n$.
   - **Semi-algbraic graphs:** Improve to n^c.
2. **Turán's Theorem.** Every $K_{2,2}$-free graph on n vertices has at most $cn^{3/2}$ edges.
   - **Semi-algbraic graphs:** Improve to $O(n^{3/2-\varepsilon})$.
3. **Szemerédi's regularity lemma.**
   - **Semi-algbraic graphs:** Quantitative and qualitative improvements.

Semi-algebraic vs VC-dimension

An application of the Milnor-Thom theorem:

Theorem

There are at most $2^{cn\log n}$ semi-algebraic graphs on n vertices and with complexity at most d, where $c = c(d)$.

Theorem (Anthony, Brightwell, Cooper 1995)

There are at least $2^{n^{2-\varepsilon}}$ graphs with VC-dimension at most d on n vertices, where $\varepsilon = \varepsilon(d)$.

Classical theorems

2. **Turán's Theorem.** Every $K_{2,2}$-free graph on n vertices has at most $cn^{3/2}$ edges.
    - **Semi-algbraic graphs:** Improve to $O(n^{3/2-\varepsilon})$.

Classical theorems

2. **Turán's Theorem.** Every $K_{2,2}$-free graph on n vertices has at most $cn^{3/2}$ edges.
   - **Semi-algbraic graphs.** Improve to $O(n^{3/2-\varepsilon})$.
   - **Bounded VC-dimension.** No improvement. There are $K_{2,2}$-free graphs on n vertices with $\Omega(n^{3/2})$ edges.

Two Ramsey-type results

In joint work with Jacob Fox and János Pach

1. We find large cliques or independent sets in graphs with bounded VC-dimension.
2. We find large monochromatic cliques in multi-colored graphs with bounded VC-dimension.

Two Ramsey-type results

In joint work with Jacob Fox and János Pach

1. We find large cliques or independent sets in graphs with bounded VC-dimension (Erdős-Hajnal conjecture).
2. We find large monochromatic cliques in multi-colored graphs with bounded VC-dimension.

The Erdős-Hajnal conjecture

Theorem (Erdős-Szekeres 1935)

Every graph on n-vertices contains a clique or and independent set of size $\Omega(\log n) \approx e^{\log \log n}$.

Theorem (Erdős-Hajnal 1989)

For every (fixed) graph H*, there is a constant* $c = c(H)$ *such that the following holds. Every graph on n-vertices that does not contain* H *as an induced subgraph contains a clique or and independent set of size* $e^{c\sqrt{\log n}}$.

Conjecture (Erdős-Hajnal): Improve this to n^c, where $c = c(H)$.

Known results

H = forbidden induced graphs. The Erdős-Hajnal conjecture holds for

1. H with at most 3 vertices.
2. H has 4 vertices (Gyárfás 1997).
3. "blow ups" (Alon-Pach-Solymosi 2001)
4. H is a bull (5 vertices, 5 edges, Chudnovsky-Safra 2008)

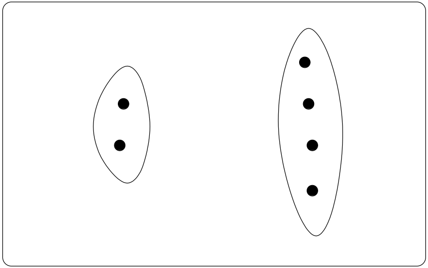
OPEN: $H = C_5$. Recently improved to $e^{c\sqrt{\log n \log\log n}}$ by Chudnovsky-Fox-Scott-Seymour-Spirkl.

Let $G = (V, E)$, $|V| = n$, with VC-dimension less than d.

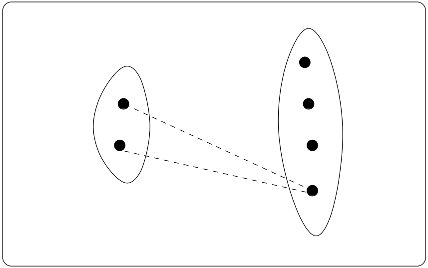
G =

Let $G = (V, E)$, $|V| = n$, with VC-dimension less than d.

G =

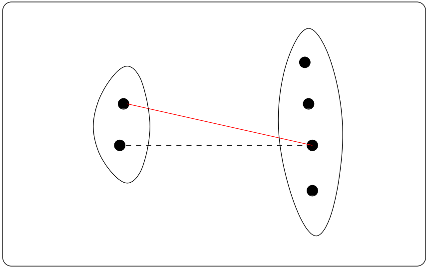


Let $G = (V, E)$, $|V| = n$, with VC-dimension less than d.

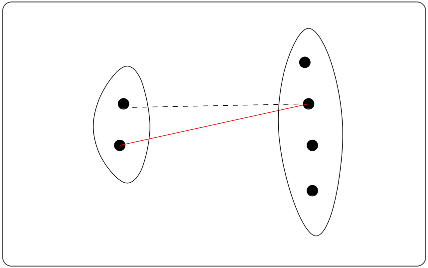
G = 

Let $G = (V, E)$, $|V| = n$, with VC-dimension less than d.

G =

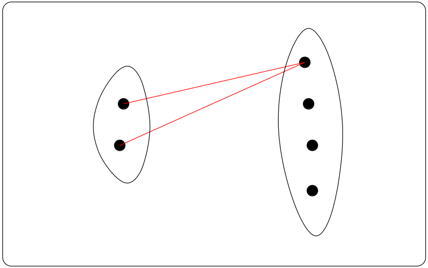


Let $G = (V, E)$, $|V| = n$, with VC-dimension less than d.

$G =$ 

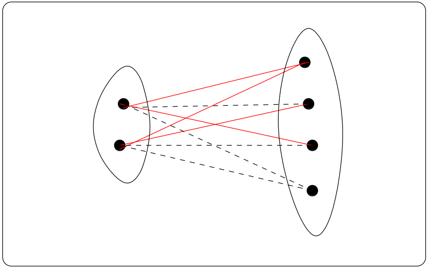
Let $G = (V, E)$, $|V| = n$, with VC-dimension less than d.

G =



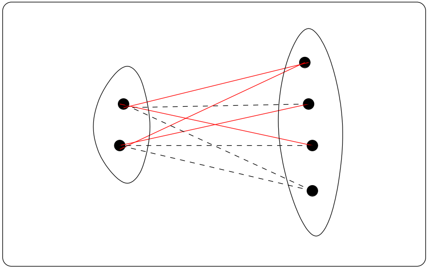
Let $G = (V, E)$, $|V| = n$, with VC-dimension less than d.

G =



Let $G = (V, E)$, $|V| = n$, with VC-dimension less than d.

G =



There is a graph H on $d + 2^d$ vertices such that H is not an induced subgraph of G.

Theorem (Erdős-Hajnal 1989)

For every (fixed) graph H, there is a constant $c = c(H)$ such that the following holds. Every graph on n-vertices that does not contain H as an induced subgraph contains a clique or and independent set of size $e^{c\sqrt{\log n}}$.

Corollary

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{c\sqrt{\log n}}$, where $c = c(d)$.

Conjecture: Improve the result above to $n^c = e^{c\log n}$, where $c = c(d)$.

First Ramsey-type result

Theorem (Erdős-Hajnal 1989)

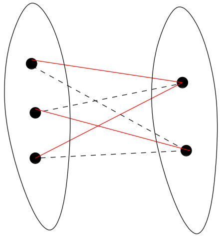
For every (fixed) graph H, there is a constant $c = c(H)$ such that the following holds. Every graph on n-vertices that does not contain H as an induced subgraph contains a clique or and independent set of size $e^{c\sqrt{\log n}}$.

Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Here, $o(1) = c\frac{\log d}{\log \log n}$.

Let H be a bipartite graph, $|V(H)| = k$.

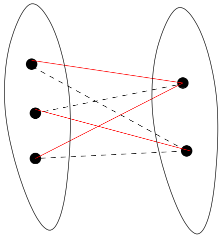
$H =$ 

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H as an induced bipartite graph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof: G has VC-dimension at most $d = d(H)$.

Let H be a bipartite graph, $|V(H)| = k$.

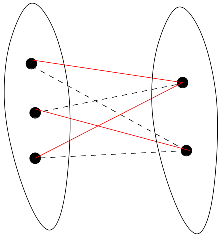
$H =$ 

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H as an induced bipartite graph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Note: We are forbidding $2^{O(k^2)}$ induced subgraphs.

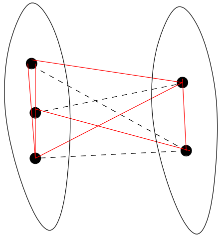
Let H be a bipartite graph, define H_1, H_2, H_3, H_4 as follows.

$H =$ 

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

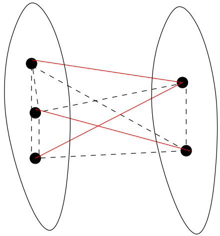
Let H be a bipartite graph, define H_1, H_2, H_3, H_4 as follows.

$H_1 =$ 

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Let H be a bipartite graph, define H_1, H_2, H_3, H_4 as follows.

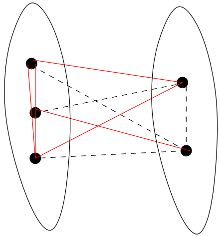
$H_2 =$ 

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Let H be a bipartite graph, define H_1, H_2, H_3, H_4 as follows.

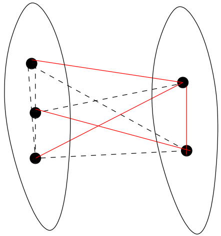
$H_3 =$



Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Let H be a bipartite graph, define H_1, H_2, H_3, H_4 as follows.

$H_4 =$ 

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof: G has VC-dimension less than $d = d(H)$.

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

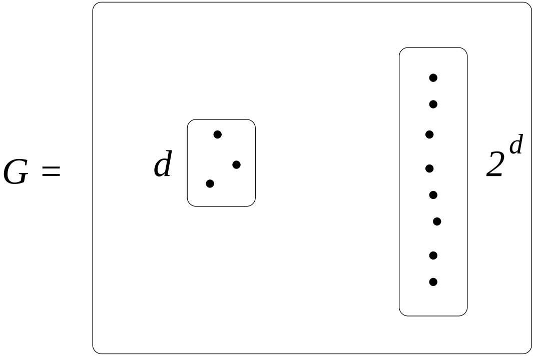
Proof: For contradiction, suppose VC-dimension is at least d.

$G =$

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

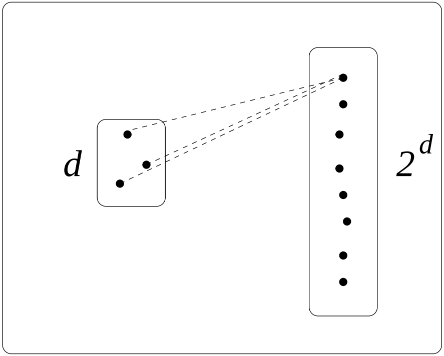
Proof: For contradiction, suppose VC-dimension is at least d.



Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

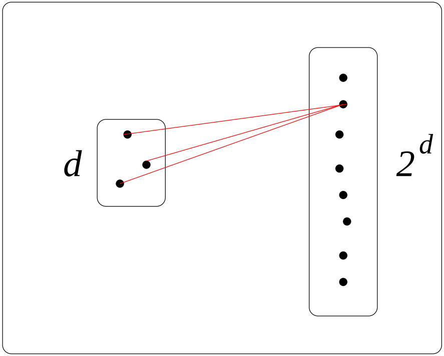
Proof: For contradiction, suppose VC-dimension is at least d.

$G =$ 

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

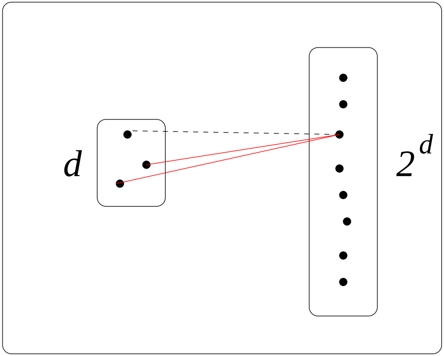
Proof: For contradiction, suppose VC-dimension is at least d.

$G =$ 

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof: For contradiction, suppose VC-dimension is at least d.

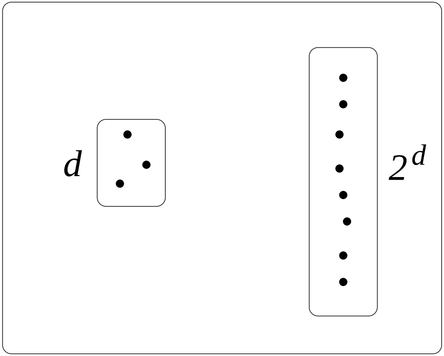
$G =$ 

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof: By the Erdős-Hajnal theorem:

$G =$

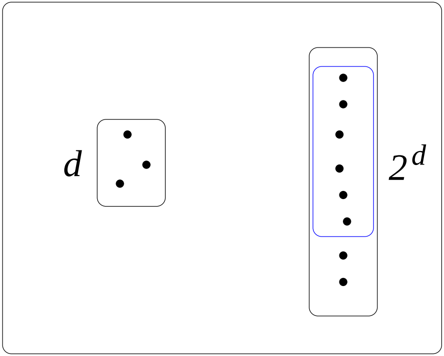


Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof: By the Erdős-Hajnal theorem:

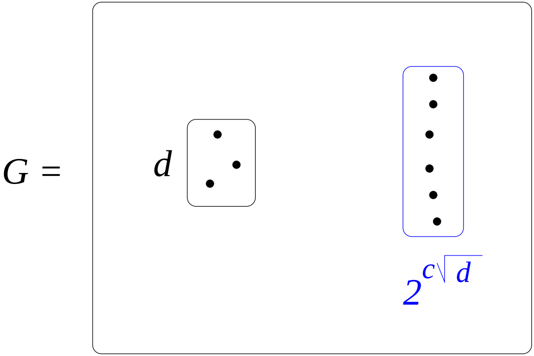
$G =$



Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

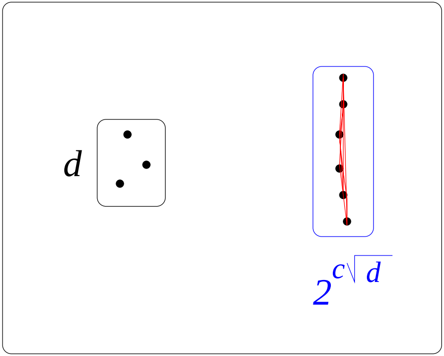
Proof: By the Erdős-Hajnal theorem:



Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

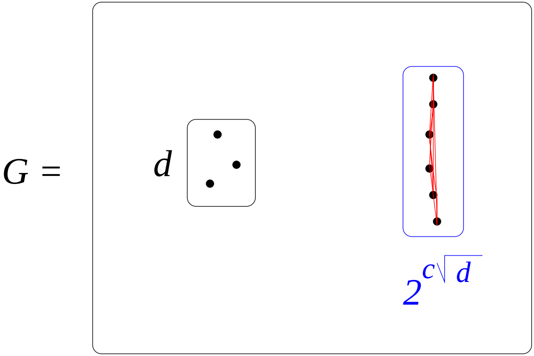
Proof: By the Erdős-Hajnal theorem:

$G =$ 

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

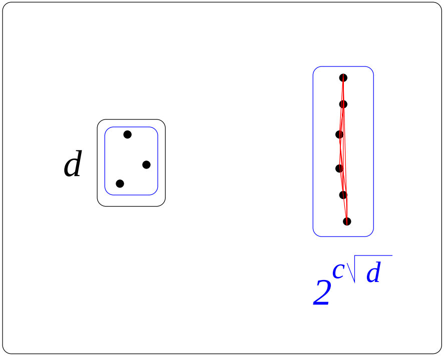
Proof: Sauer-Shelah: $|\mathcal{F}| > c|V|^k$ implies k-set shattered.



Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

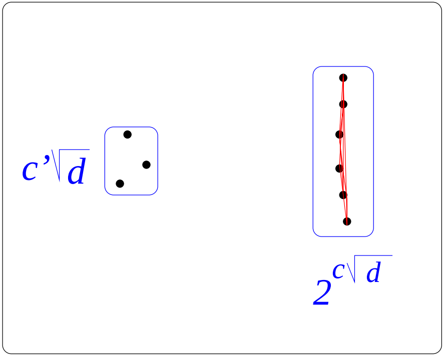
Proof: Sauer-Shelah: $|\mathcal{F}| > c|V|^k$ implies k-set shattered.

$G =$ 

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

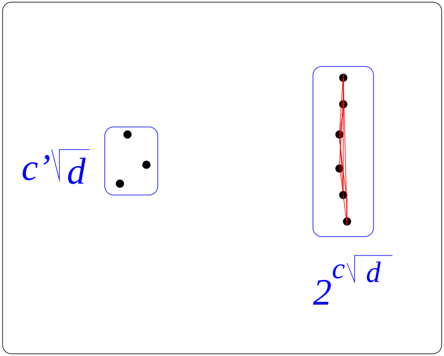
Proof: $c'\sqrt{d}$ vertices are shattered.

$G =$ 

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof: By Erdős-Hajnal (again)

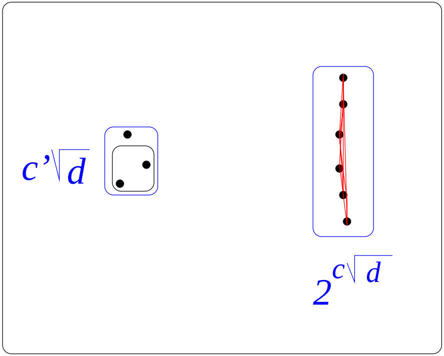
$G =$ 

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof: By Erdős-Hajnal (again)

$G =$

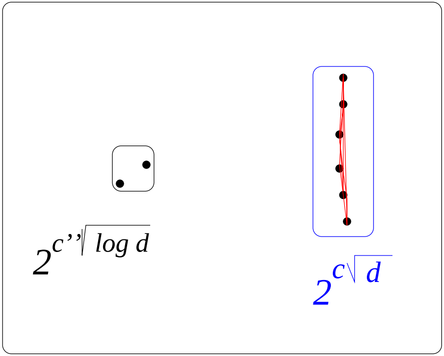


Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof: By Erdős-Hajnal (again)

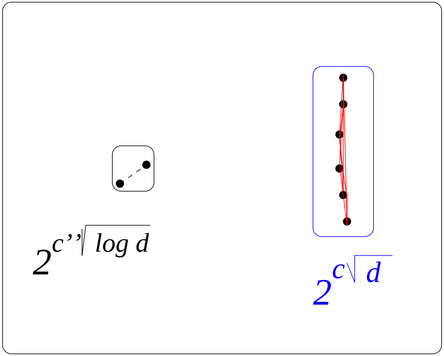
$G =$



Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof: By Erdős-Hajnal (again)

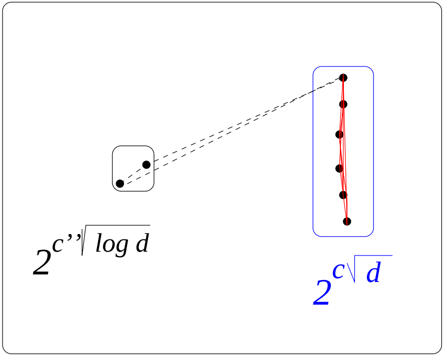
$G =$ 

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof: The $2^{c'' \log d}$ set is shattered.

$G =$

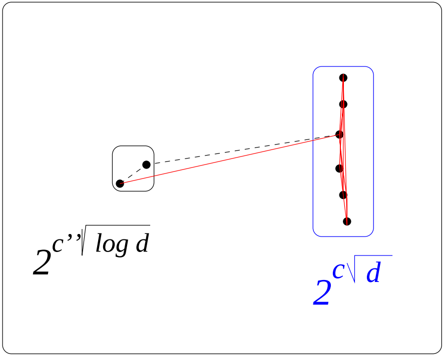


Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof: The $2^{c'' \log d}$ set is shattered.

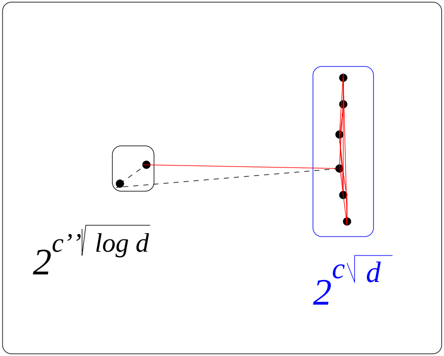
$G =$



Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof: The $2^{c'' \log d}$ set is shattered.

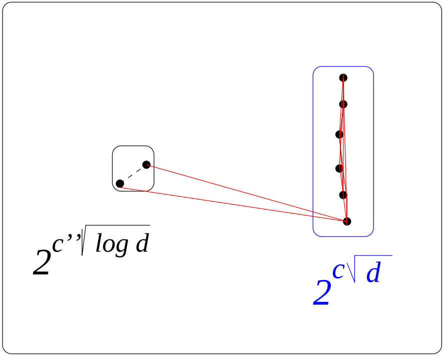
$G =$ 

Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof: The $2^{c'' \log d}$ set is shattered.

$G =$

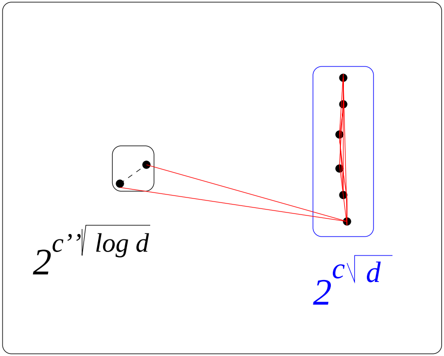


Corollary (Fox-Pach-S. 2019)

Let H be a fixed bipartite graph. Then every n-vertex graph that does not contain H_1, H_2, H_3, H_4 as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof: For large $d = d(H)$, we obtain H_1, H_2, H_3, or H_4. □

$G =$



Main result

Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

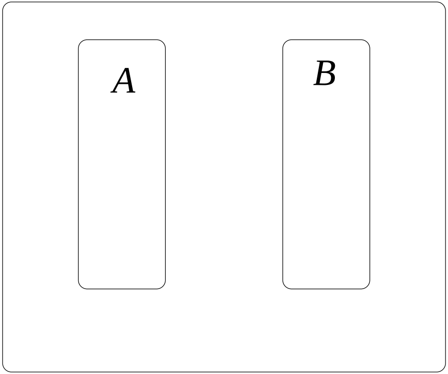
Proof idea. (Erdős-Hajnal) G is H induced free. Use induction to find a large perfect subgraph.

G =

Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

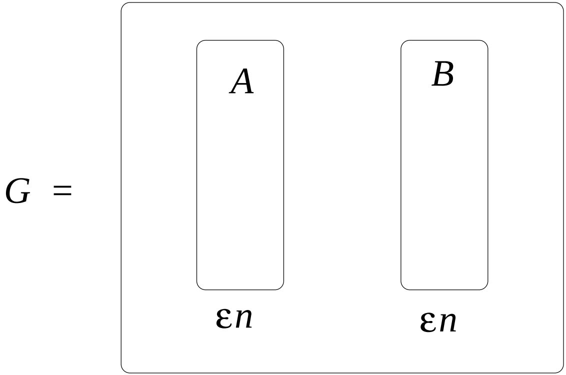
Proof idea. (Erdős-Hajnal) Lemma: $\exists A, B \subset V$

$G =$ 

Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

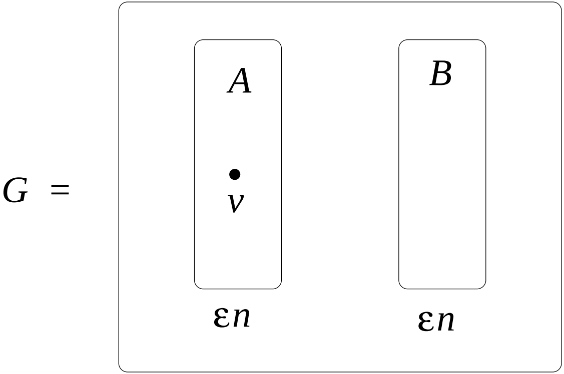
Proof idea. (Erdős-Hajnal) Lemma: $\exists A, B \subset V$



Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

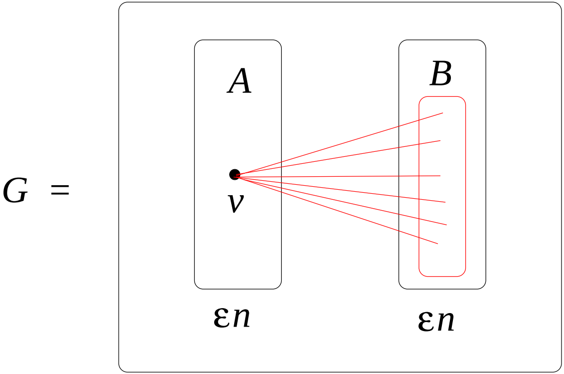
Proof idea. (Erdős-Hajnal) Lemma: $\exists A, B \subset V$



Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

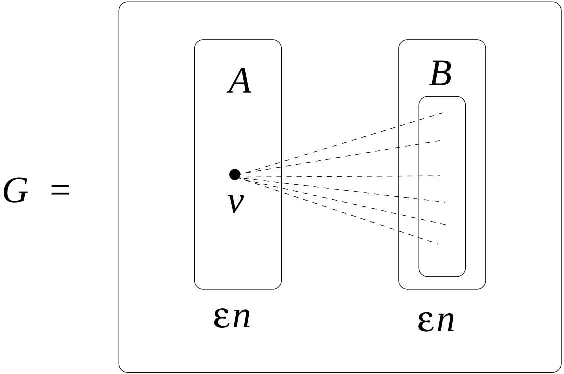
Proof idea. (Erdős-Hajnal) $v \in A$, $|N(v) \cap B| > (1-\varepsilon)|B|$ or $|N(v) \cap B| < \varepsilon|B|$



Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

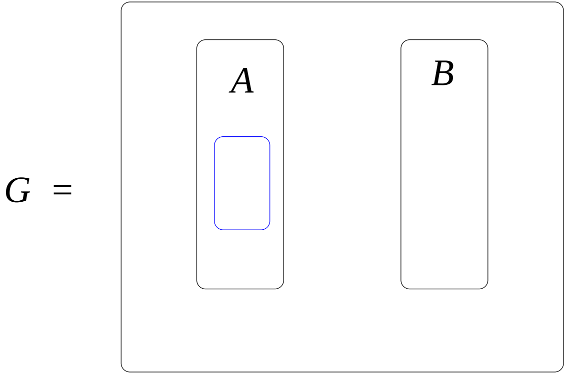
Proof idea. (Erdős-Hajnal) $v \in A$, $|N(v) \cap B| > (1-\varepsilon)|B|$ or $|N(v) \cap B| < \varepsilon|B|$



Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

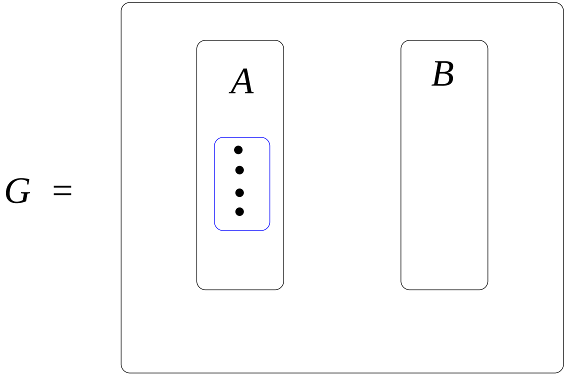
Proof idea. (Erdős-Hajnal) Apply induction in A.



Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

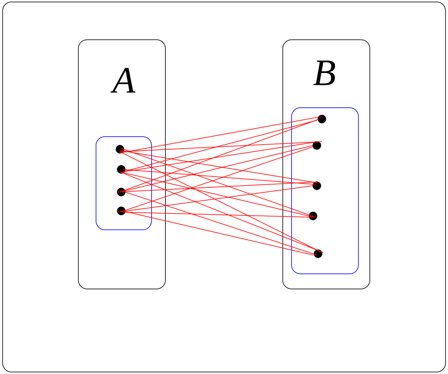
Proof idea. (Erdős-Hajnal) Apply induction in A.



Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

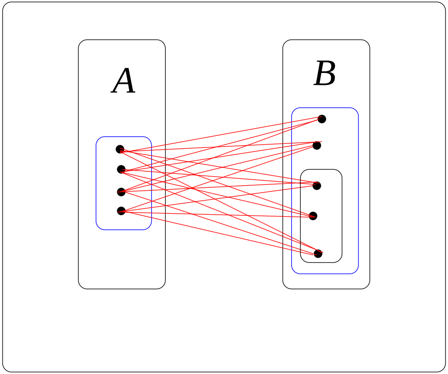
Proof idea. (Erdős-Hajnal) Apply induction in A.

$G =$ 

Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. (Erdős-Hajnal) Apply induction in B.

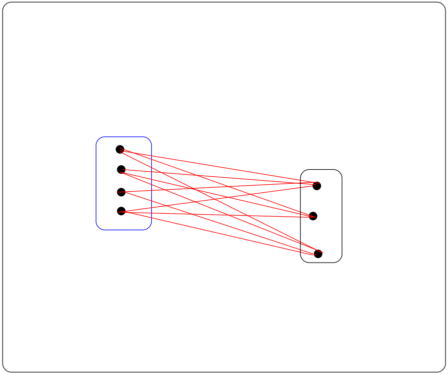
$G =$ 

Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. (Erdős-Hajnal) Apply induction in B. Combine to get a large perfect graph.

G =



Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. G has bounded VC-dimension.

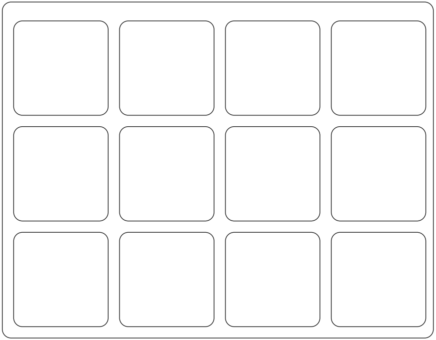
$$G =$$

Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. Apply a strong regularity lemma.

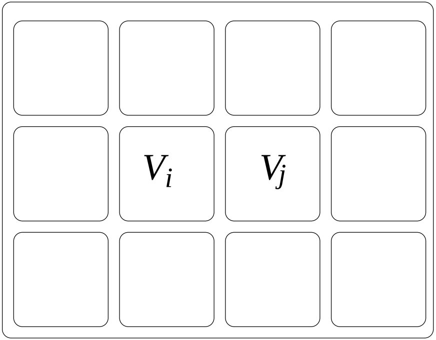
$G =$



Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. Apply a strong regularity lemma.

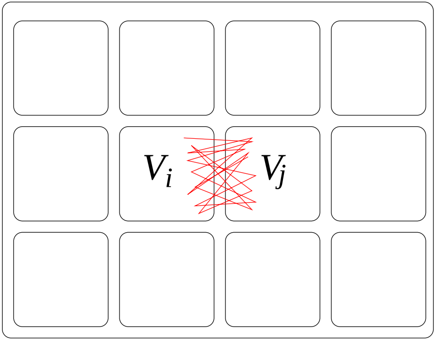
$G =$ 

Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. At least $(1-\varepsilon)|V_i||V_j|$ edges

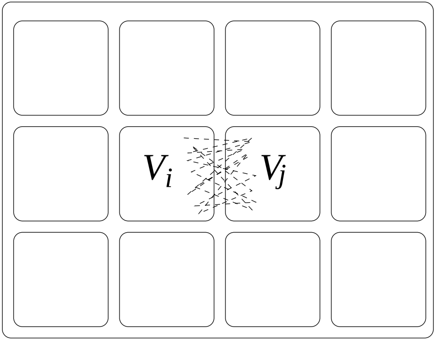
$G =$



Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

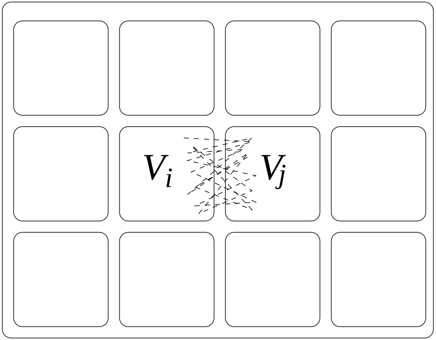
Proof idea. Less than $\varepsilon|V_i||V_j|$ edges

$G =$ 

Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. True for all but an ε-fraction pairs of parts.

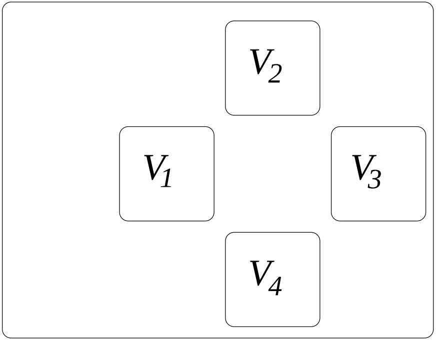
$G =$ 

Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. Apply Turan's theorem

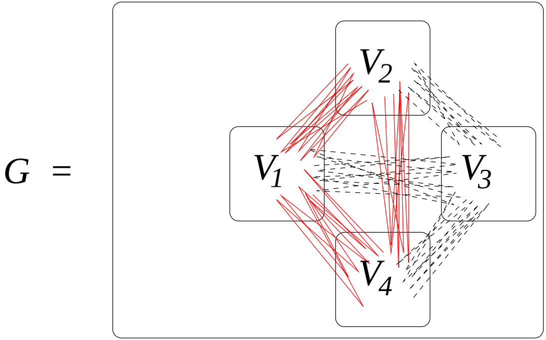
$G =$



Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

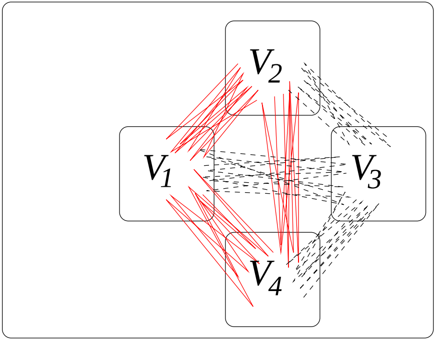
Proof idea. Apply Turan's theorem



Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

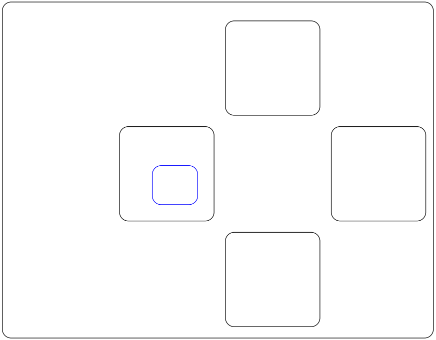
Proof idea. Follow the Erdős-Hajnal argument with many parts.

$G =$ 

Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

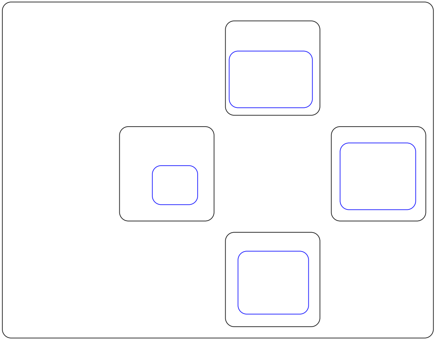
Proof idea. Apply induction inside one of the parts.

$G =$ 

Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

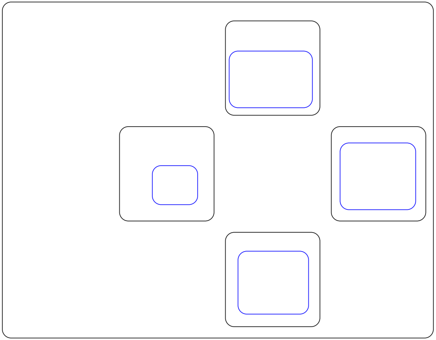
Proof idea. Complete or empty bipartite graphs.

$G =$ 

Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. Complete or empty bipartite graphs.

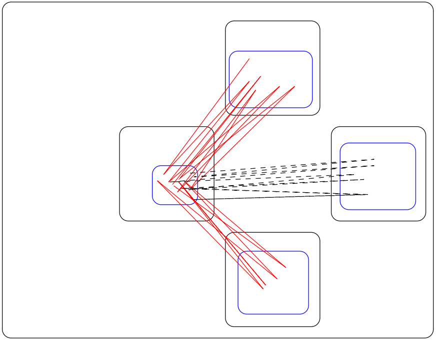
$G =$ 

Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. Complete or empty bipartite graphs.

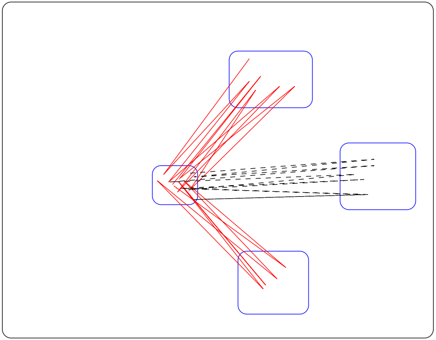
G =



Theorem (Fox-Pach-S. 2019)

For every n-vertex graph with VC-dimension at most d contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. Repeat the argument.

$G =$ 

Two Ramsey-type results

In joint work with Jacob Fox and János Pach

1. We find large cliques or independent sets in graphs with bounded VC-dimension (Erdős-Hajnal conjecture).
2. We find large monochromatic cliques in multi-colored graphs with bounded VC-dimension.

Two Ramsey-type results

In joint work with Jacob Fox and János Pach

1. We find large cliques or independent sets in graphs with bounded VC-dimension (Erdős-Hajnal conjecture).
2. We find large monochromatic cliques in multi-colored graphs with bounded VC-dimension (Schur-Erdős conjecture).

Multicolor Ramsey numbers

Color all edges of K_N with m colors.

Question: How large does N have to be to guarantee a monochromatic K_3?

$K_N =$

Multicolor Ramsey numbers

Color all edges of K_N with m colors.

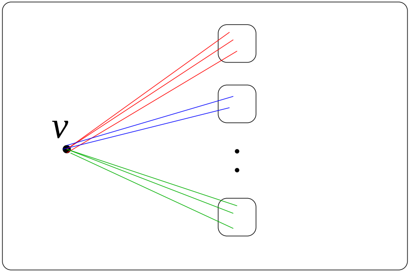
Question: How large does N have to be to guarantee a monochromatic K_3?

$K_N =$

Multicolor Ramsey numbers

Color all edges of K_N with m colors.

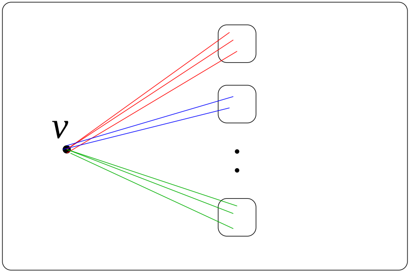
Question: How large does N have to be to guarantee a monochromatic K_3?

$K_N =$ 

Multicolor Ramsey numbers

Color all edges of K_N with m colors.

Question: How large does N have to be to guarantee a monochromatic K_3?

$K_N =$ 

Notation: $N_i(v) = \{u \in V : \chi(uv) = i\}$.

Multicolor Ramsey numbers

Definition

For $m \geq 2$, The multicolor Ramsey number

$$r(\underbrace{3, \ldots, 3}_{m \text{ times}})$$

is the minimum integer N such that for any m-coloring of the edges of K_N contains a monochromatic copy of K_3.

Known results

$$r(3,3) = 6 \qquad r(3,3,3) = 17 \qquad 51 \leq r(3,3,3,3) \leq 62$$

$$162 \leq r(3,3,3,3,3) \leq 307$$

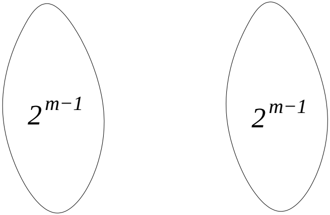
$$2^m < r(\underbrace{3,\ldots,3}_{m \text{ times}}) < m!$$

Known results

$$r(3,3) = 6 \qquad r(3,3,3) = 17 \qquad 51 \leq r(3,3,3,3) \leq 62$$

$$162 \leq r(3,3,3,3,3) \leq 307$$

$$2^m < r(\underbrace{3,\ldots,3}_{m \text{ times}}) < m!$$

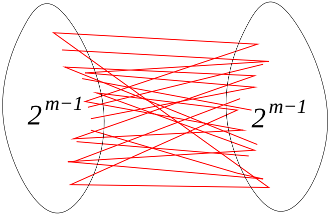


Known results

$$r(3,3) = 6 \qquad r(3,3,3) = 17 \qquad 51 \leq r(3,3,3,3) \leq 62$$

$$162 \leq r(3,3,3,3,3) \leq 307$$

$$2^m < r(\underbrace{3,\ldots,3}_{m \text{ times}}) < m!$$

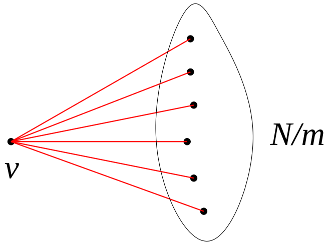


Known results

$$r(3,3) = 6 \qquad r(3,3,3) = 17 \qquad 51 \leq r(3,3,3,3) \leq 62$$

$$162 \leq r(3,3,3,3,3) \leq 307$$

$$2^m < r(\underbrace{3,\ldots,3}_{m \text{ times}}) < m!$$



Best known bounds

Lower bound: Fredricksen-Sweet, Abbot-Moser.

Upper bound: Schur.

$$(3.199)^m < r(\underbrace{3, \ldots, 3}_{m \text{ times}}) < 2^{O(m \log m)}$$

Erdős prize problems

Problem (\$100)

Is the limit below finite or infinite?

$$\lim_{m\to\infty}\left(r(\underbrace{3,\ldots,3}_{m \text{ times}})\right)^{1/m}$$

Problem (\$250)

Determine

$$\lim_{m\to\infty}\left(r(\underbrace{3,\ldots,3}_{m \text{ times}})\right)^{1/m}$$

Main result

If we insist that the m-coloring has bounded VC-dimension:

$$\mathcal{F} = \{N_i(v) : v \in V, i \in [m]\}$$

$\mathcal{F}$ has VC-dimension at most $d = O(1)$.

Theorem (Fox-Pach-S. 2020)

For $m \geq 2$,

$$r_d(\underbrace{3, \ldots, 3}_{m \text{ times}}) = 2^{\Theta(m)}.$$

Main result

If we insist that the m-coloring has bounded VC-dimension:

$$\mathcal{F} = \{N_i(v) : v \in V, i \in [m]\}$$

$\mathcal{F}$ has VC-dimension at most $d = O(1)$.

Theorem (Fox-Pach-S. 2020)

For fixed $p \geq 3$ and $m \geq 2$,

$$r_d(\underbrace{p, \ldots, p}_{m \text{ times}}) = 2^{\Theta(m)}.$$

Main result

If we insist that the m-coloring has bounded VC-dimension:

$$\mathcal{F} = \{N_i(v) : v \in V, i \in [m]\}$$

$\mathcal{F}$ has VC-dimension at most $d = O(1)$.

Theorem (Fox-Pach-S. 2020)

For fixed $p \geq 3$ and $m \geq 2$,

$$r_d(\underbrace{p, \ldots, p}_{m \text{ times}}) = 2^{\Theta(m)}.$$

Proof idea: Use a different partition result based on Haussler's packing lemma.

Open Problems

1. Is the Erdős-Hajnal conjecture true for string graphs.
2. Find more results for graphs with bounded VC dimension.

Thank you!