Bounded VC-dimension implies the Schur-Erdos conjecture

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June 24, 2020

Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

Definition

A set $S \subset V$ is **shattered** by $\mathcal{F}$ if for all $X \subset S$, there is an $A \in \mathcal{F}$ such that $S \cap A = X$.

Definition

The **VC-dimension of** $\mathcal{F}$ is the size of the largest subset $S \subset V$ that is shattered by $\mathcal{F}$.

VC-dimension of a graph

$G = (V, E)$, let $\mathcal{F} \subset 2^V$ such that $\mathcal{F} = \{N(v) : v \in V\}$.
$|V| = |\mathcal{F}| = n$

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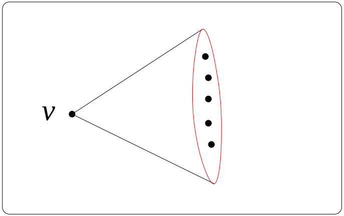
$G =$ 

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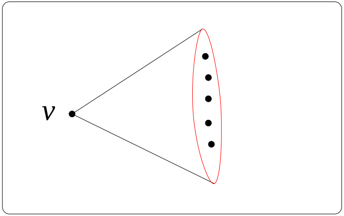
$G =$ 

Definition

The VC-dimension of G is the VC-dimension of $\mathcal{F}$.

Examples of graphs with bounded VC-dimension

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Problem

Can we substantially improve some of the classical theorems in extremal graph theory for graphs with bounded VC-dimension?

Classical theorems

1. **Ramsey's Theorem.** Every graph on n vertices contains a clique or independent set of size $c \log n$.

2. **Turán's Theorem.** Every $K_{2,2}$-free graph on n vertices has at most $cn^{3/2}$ edges.

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Can we improve these classical results for graphs with bounded VC-dimension?

Semi-algebraic vs VC-dimension

An application of the Milnor-Thom theorem:

Theorem

There are at most $2^{cn\log n}$ semi-algebraic graphs on n vertices and with complexity at most d, where $c = c(d)$.

Theorem (Anthony, Brightwell, Cooper 1995)

There are at least $2^{n^{2-\varepsilon}}$ graphs with VC-dimension at most d on n vertices, where $\varepsilon = \varepsilon(d)$.

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Classical theorems

2. **Turán's Theorem.** Every $K_{2,2}$-free graph on n vertices has at most $cn^{3/2}$ edges.
   - **Semi-algebraic graphs.** Improve to $O(n^{3/2-\varepsilon})$.
   - **Bounded VC-dimension.** No improvement. There are $K_{2,2}$-free graphs on n vertices with $\Omega(n^{3/2})$ edges.

Main result

In joint work with Jacob Fox and János Pach

- We establish tight bounds for multicolor Ramsey numbers for graphs with bounded VC-dimension.

Multicolor Ramsey numbers

Definition

For $m \geq 2$, The multicolor Ramsey number

$$r(\underbrace{3,\ldots,3}_{m \text{ times}})$$

is the minimum integer N such that for any m-coloring of the edges of K_N contains a monochromatic copy of K_3.

Known results

$$r(3,3) = 6 \qquad r(3,3,3) = 17 \qquad 51 \leq r(3,3,3,3) \leq 62$$

$$162 \leq r(3,3,3,3,3) \leq 307$$

$$2^m < r(\underbrace{3,\ldots,3}_{m \text{ times}}) < m!$$

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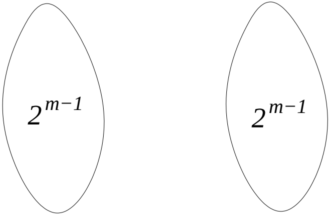

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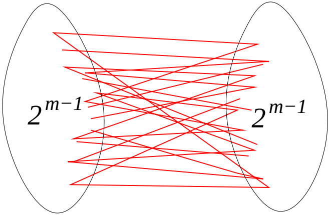

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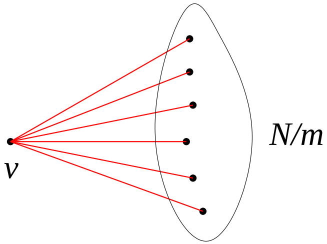

Best known bounds

Lower bound: Fredricksen-Sweet, Abbot-Moser.

Upper bound: Schur.

$$(3.199)^m < r(\underbrace{3,\ldots,3}_{m \text{ times}}) < 2^{O(m\log m)}$$

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Conjecture (Schur-Erdős)

$$r(\underbrace{3, \ldots, 3}_{m \text{ times}}) = 2^{\Theta(m)}.$$

Bounded VC-dimension setting

Color all edges of K_N with m colors.

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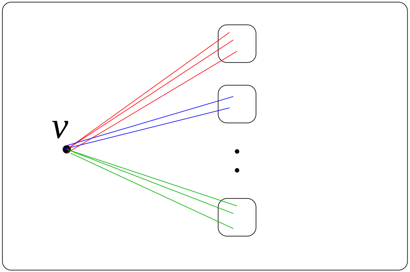

Notation: $N_i(v) = \{u \in V : \chi(uv) = i\}$.

Main result

If we insist that the m-coloring has bounded VC-dimension:

$$\mathcal{F} = \{N_i(v) : v \in V, i \in [m]\}$$

$\mathcal{F}$ has VC-dimension at most $d = O(1)$.

Theorem (Fox-Pach-S. 2020)

For $m \geq 2$,

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Sketch of the proof:

$$r_d(\underbrace{3, \ldots, 3}_{m \text{ times}}) \leq 2^{cm}, \qquad c = c(d)$$

Idea: We will use induction on m. Set $N = 2^{cm}$ and let V be an N-element vertex set.

$\chi : \binom{V}{2} \to \{1, 2, \ldots, m\}$ and $\mathcal{F} = \{N_i(v) : v \in V, i \in [m]\}$.

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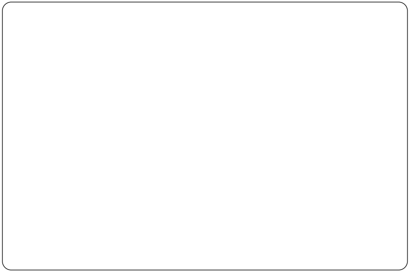

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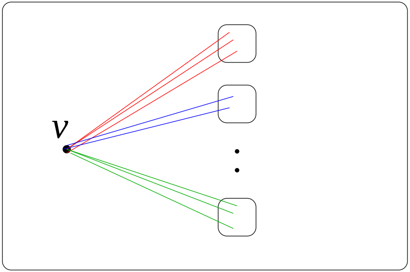
$K_N =$ 

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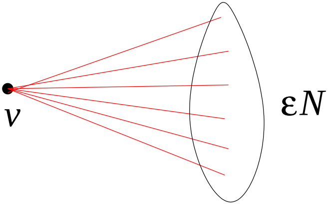

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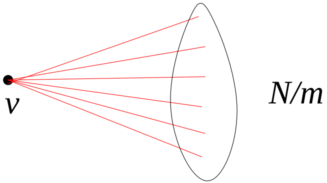
Not true: We can only assume $|N_i(v)| \geq N/m$ by pigeonhole.



Crossing pairs of vertices

$\mathcal{F} = \{N_i(v) : v \in V, i \in [m]\}$.

Crossing: Let $A \in \mathcal{F}$ and $u, v \in V$. Then A crosses $\{u, v\}$ if it contains one but not the other.

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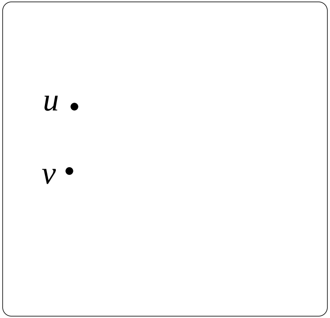

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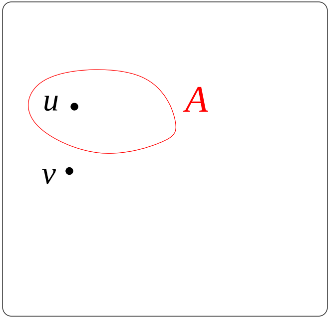
$K_N =$ 

Partition Lemma

$\mathcal{F} = \{N_i(v) : v \in V, i \in [m]\}$, dual VC-dimension d.

Lemma

For any δ satisfying $1 \leq \delta \leq |\mathcal{F}|$, there is an equipartition $V = S_1 \cup \cdots \cup S_r$ of V into $r \leq c(|\mathcal{F}|/\delta)^d$ parts, such that any pair of vertices from the same part S_t is crossed by at most 2δ members of $\mathcal{F}$.

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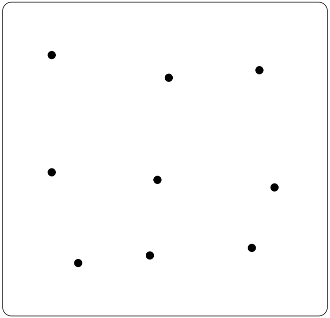

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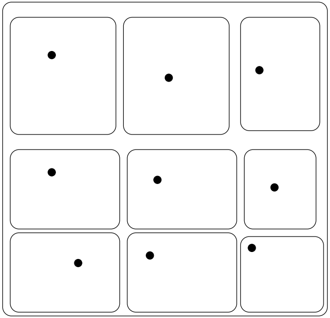
$K_N =$ 

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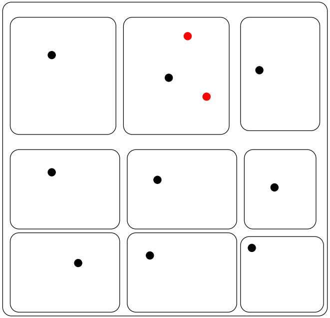
$K_N =$ 

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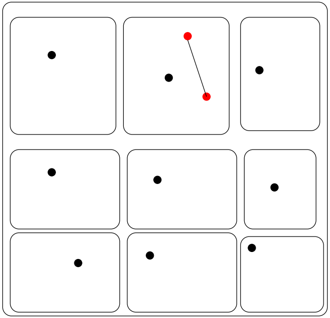

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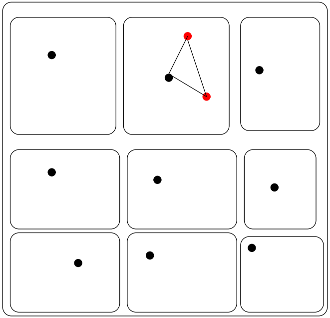

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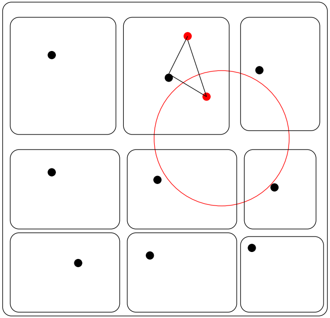
$K_N =$ 

Key observation:

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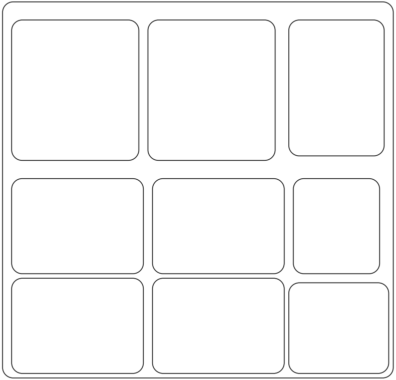

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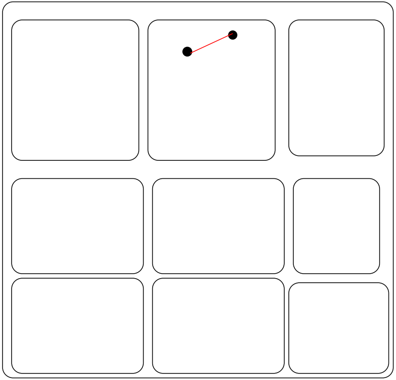
$K_N =$ 

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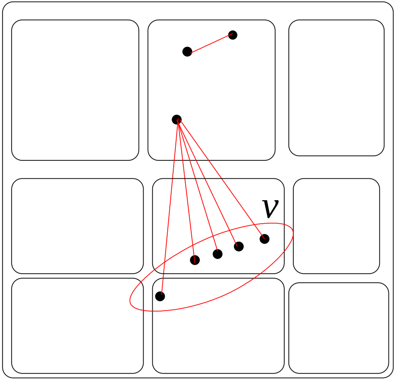
$K_N =$ 

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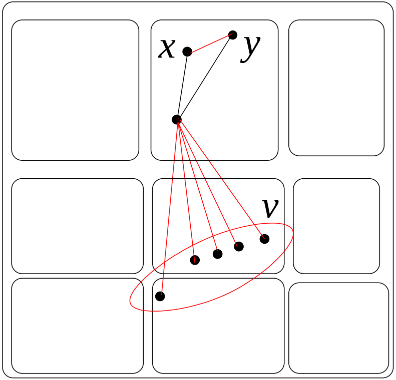
$K_N =$ 

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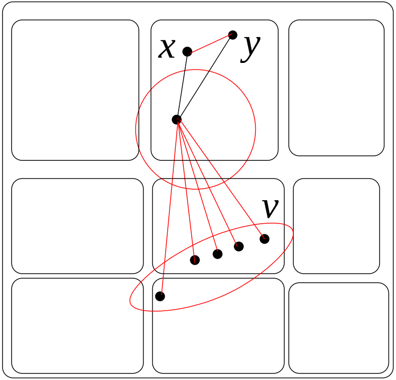

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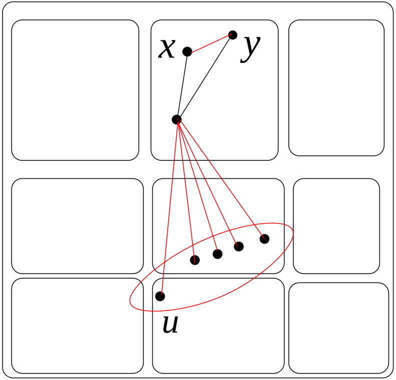
$K_N =$ 

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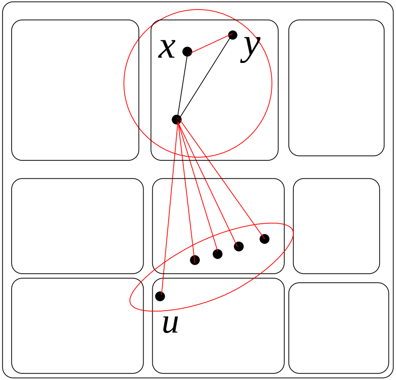

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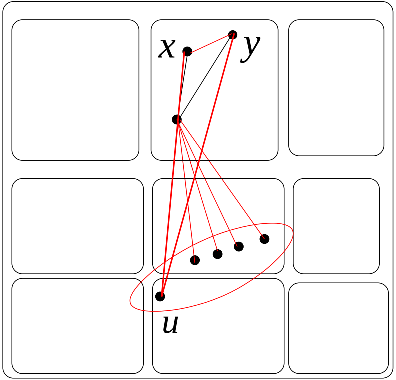

Case 1. If a part is missing many colors, we are done by induction.

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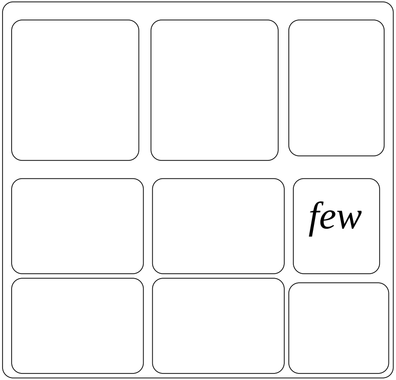

Partition Lemma

Case 2. Each part has many distinct colors

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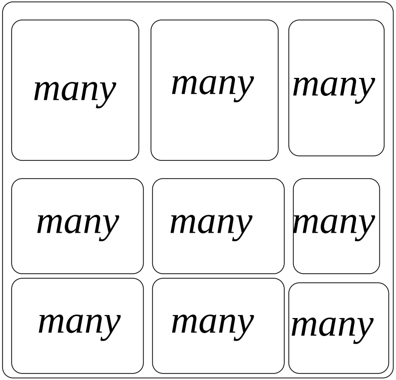

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Key observation: Otherwise, many vertices have small neighborhoods with respect to some colors.

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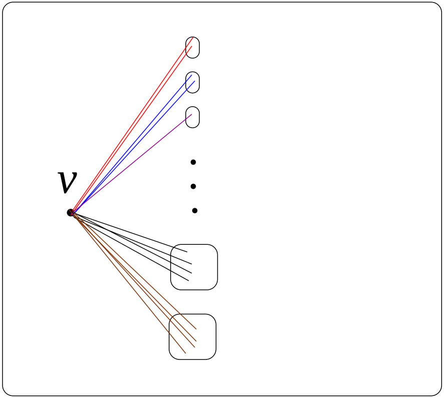

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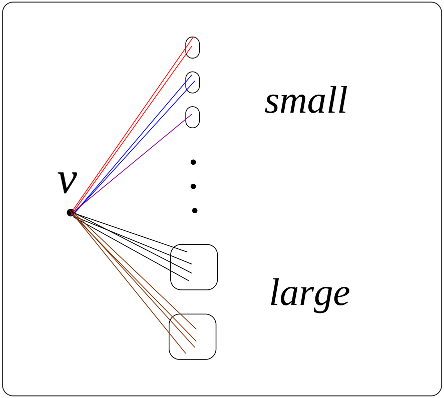

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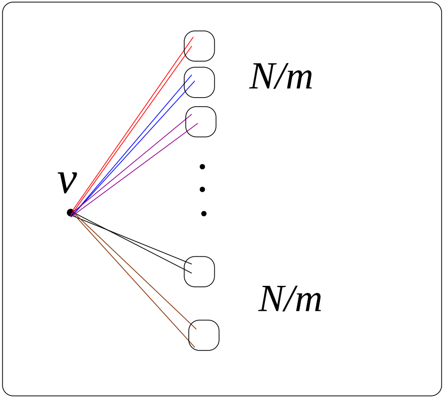

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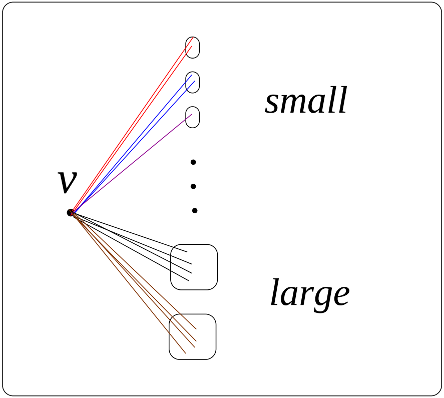

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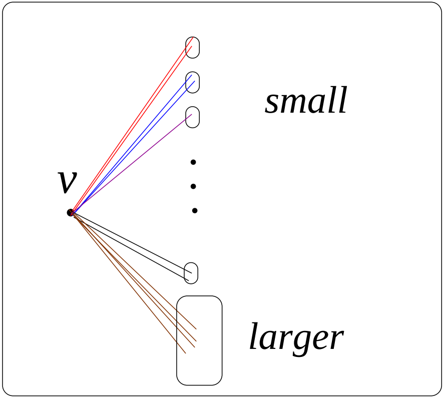
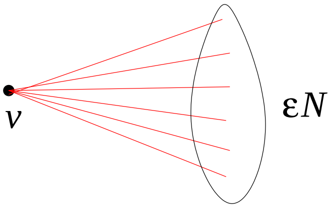


Partition Lemma

Goal: Find a vertex with large degree with respect to one color class.



Open Problem

Erdős-Hajnal conjecture:

Problem

Given a graph with bounded VC-dimension, does it contain a clique or independent set of size n^{ε}?

- Best known bound is $e^{(\log n)^{1-o(1)}}$, where $o(1) \approx \frac{\log d}{\log \log n}$ (Fox-Pach-S.).

Thank you!