Ramsey numbers and an old problem of Erdős and Hajnal

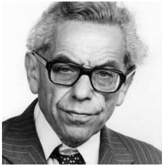
Andrew Suk (UC San Diego)

February 9, 2022

Origins of Ramsey theory

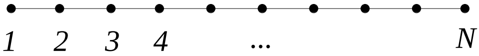
Frank Ramsey (1928)

Paul Erdős and George Szekeres (1935)



Ramsey theory

Formal definition: For any integers $k \geq 1$, $s, n \geq k$, there is a minimum $r_k(s, n) = N$, such that for every red/blue coloring of the k-tuples of $\{1, 2, \ldots, N\}$,



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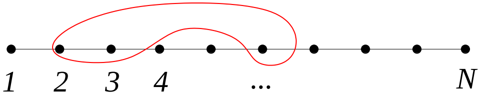

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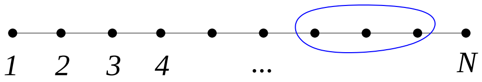

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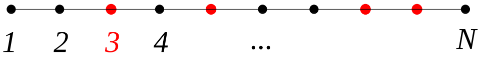

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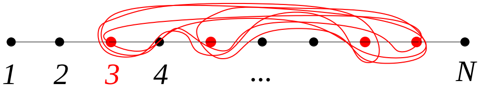

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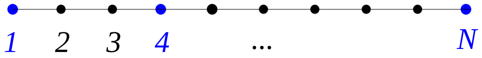

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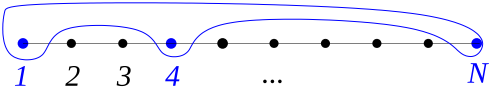

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$r_k(s,n) =$ Ramsey numbers

Graph Ramsey theorem

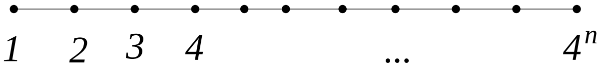
Theorem (Erdős-Szekeres 1935)

$$r_2(s,n) \leq \binom{n+s-2}{s-1}$$

$$r_2(n,n) \leq \binom{2n-2}{n-1} \approx \frac{4^n}{\sqrt{n}}$$

Graph Ramsey number $r_2(n, n) \leq 4^n$

Let G be the complete graph with 4^n vertices, every edge has color red or blue.



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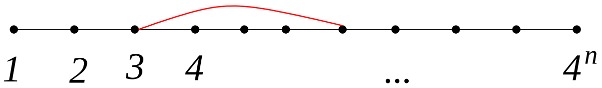

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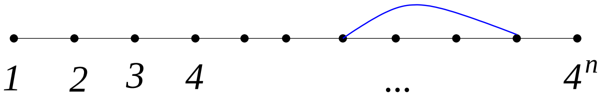

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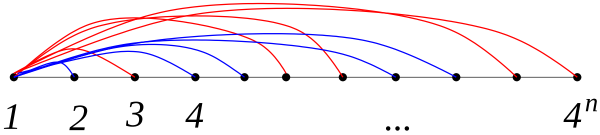

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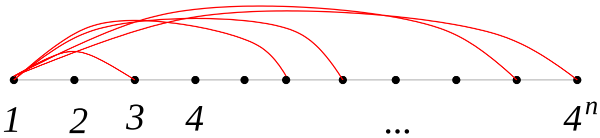

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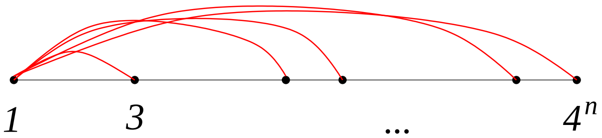

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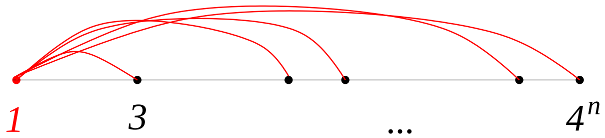

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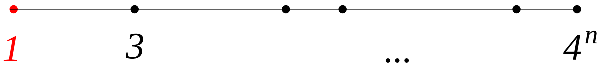

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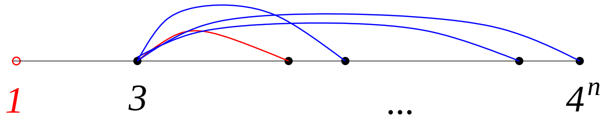

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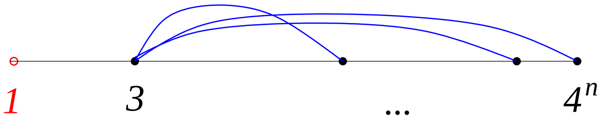

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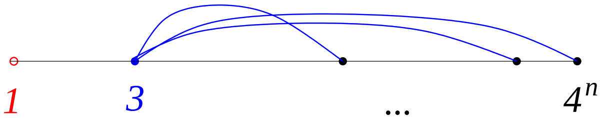

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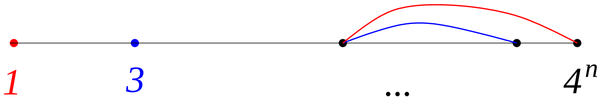

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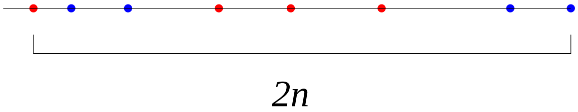

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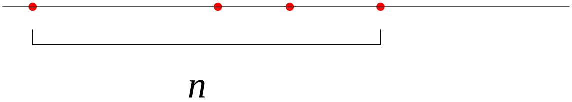

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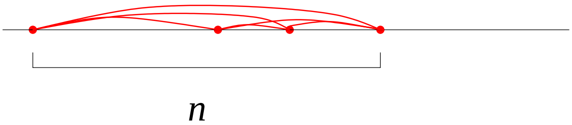


Diagonal graph Ramsey numbers

Theorem (Erdős 1947, Erdős-Szekeres 1935)

$$(1 + o(1))\frac{n}{e}2^{n/2} < r_2(n, n) < \frac{4^n}{\sqrt{n}}.$$

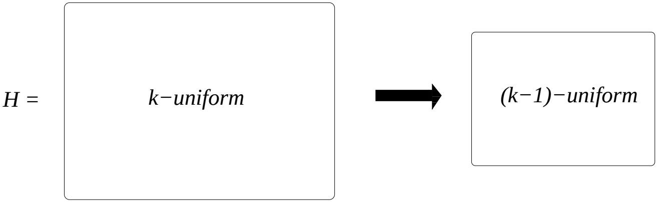
Theorem (Spencer 1977, Sah 2020)

$$(1 + o(1))\frac{\sqrt{2}}{e}n2^{n/2} < r_2(n, n) < \frac{4^n}{n^{c \log n}}$$

Upper bounds for $r_k(n, n)$

Generalize the greedy argument by considering edges emanating out of $(k-1)$-tuples.

Erdős-Rado upper bound argument



Greedy argument to reduce the problem to a $(k-1)$-uniform hypergraph problem. Argument shows

$$r_k(s,n) \leq 2^{(r_{k-1}(s-1,n-1))^{k-1}}.$$

Upper bounds for diagonal hypergraph Ramsey numbers

Applying $r_k(s,n) \leq 2^{(r_{k-1}(s-1,n-1))^{k-1}}$.

Sah (2020): $r_2(n,n) < \frac{4^n}{n^{c \log n}}$

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Erdős-Rado (1952): $r_k(n,n) < \left. 2^{\cdot^{\cdot^{2^{cn}}}} \right\} k$

Lower bounds for diagonal hypergraph Ramsey numbers

Random constructions

Spencer (1977): $\frac{\sqrt{2}}{e} n 2^{n/2} < r_2(n,n) < \frac{4^n}{n^{c \log n / \log\log n}}$

Erdős (1947): $2^{c' n^2} < r_3(n,n) < 2^{2^{cn}}$

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Lower bounds for diagonal hypergraph Ramsey numbers

Erdős-Hajnal stepping up lemma: $k \geq 3$, $r_{k+1}(n,n) > 2^{r_k(n/4,n/4)}$.

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Conjecture: $r_3(n, n) > 2^{2^{cn}}$

Theorem (Erdős-Hajnal-Rado 1952/1965)

$$2^{cn^2} < r_3(n, n) < 2^{2^{c'n}}$$

Conjecture (Erdős, $500)

$$r_3(n, n) > 2^{2^{cn}}$$

Theorem (Erdős-Hajnal)

$$r_3(n, n, n, n) > 2^{2^{cn}}$$

Off-diagonal Ramsey numbers

$r_k(s, n)$ where s is fixed, and $n \to \infty$.

Graphs:

Theorem (Ajtai-Komlós-Szemerédi 1980, Kim 1995)

$r_2(3, n) = \Theta\left(\frac{n^2}{\log n}\right)$

Theorem

For fixed $s > 3$

$$n^{(s+1)/2+o(1)} < r_2(s, n) < n^{s-1+o(1)}$$

Upper bounds for off-diagonal Ramsey numbers

3-uniform hypergraphs:

Theorem (Erdős-Hajnal-Rado)

For fixed $s \geq 4$,

$$2^{csn} < r_3(s,n) < 2^{c' n^{2s-4}}.$$

Theorem (Conlon-Fox-Sudakov 2010)

For fixed $s \geq 4$,

$$2^{csn \log n} < r_3(s,n) < 2^{c' n^{s-2} \log n}.$$

Upper bounds for off-diagonal hypergraph Ramsey numbers

Fixed $s \geq k+1$.

Erdős-Szekeres (1935): $r_2(s,n) < n^{s-1+o(1)}$

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Erdős-Rado: $r_k(s,n) < \left. 2^{\cdot^{\cdot^{2^{(n^c)}}}} \right\} k-1$

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Bohman-Keevash (2010): $n^{(s+1)/2+o(1)} < r_2(s,n) < n^{s-1+o(1)}$

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MS and CFS (2015): $2^{2^{2^{c' n}}} < r_5(8,n) < 2^{2^{2^{n^c}}}$

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What is the tower growth rate of $r_k(k+1,n)$ and $r_k(k+2,n)$?

Fixed $s \geq k+1$.

Bohman-Keevash (2010): $n^{(s+1)/2+o(1)} < r_2(s,n) < n^{s-1+o(1)}$

Conlon-Fox-Sudakov (2010): $2^{c' n \log n} < r_3(s,n) < 2^{cn^{s-2} \log n}$

Erdős-Hajnal: $2^{2^{c' n}} < r_4(7,n) < 2^{2^{n^c}}$

MS and CFS: $2^{2^{2^{c' n}}} < r_5(8,n) < 2^{2^{2^{n^c}}}$

$\vdots$

MS and CFS: $k-1\left\{ 2^{\cdot^{\cdot^{2^{c' n}}}} \right. < r_k(k+3,n) < \left. 2^{\cdot^{\cdot^{2^{(n^c)}}}} \right\} k-1$

What is the tower growth rate of $r_4(5,n)$ and $r_4(6,n)$?

Off-diagonal hypergraph Ramsey numbers

Theorem (Mubayi-S., 2018)

$$r_4(5,n) > 2^{n^{c\log n}} \qquad r_4(6,n) > 2^{2^{cn^{1/5}}}.$$

Fixed $k > 4$,

$$r_k(k+2,n) = \left. 2^{\cdot^{\cdot^{2^{(n^{\Theta(1)})}}}} \right\} k-1$$

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Open problems

Diagonal Ramsey problem ($500 Erdős):

$$2^{cn^2} < r_3(n,n) < 2^{2^{cn}}.$$

Off-diagonal Ramsey problem:

$$2^{n^{c\log n}} < r_4(5,n) < 2^{2^{cn}}.$$

Theorem (Mubayi-S. 2017)

Showing $r_3(n,n) > 2^{2^{cn}}$ implies that $r_4(5,n) > 2^{2^{c'n}}$.

More off-diagonal?

Off diagonal hypergraph Ramsey numbers: $r_k(k+1, n)$

Red clique size $k+1$ or Blue clique of size n.

$$r_k(k, n) = n \qquad \text{(trivial)}$$

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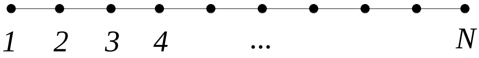
$$r_k(k, n) = n \qquad \text{(trivial)}$$

A more off diagonal Ramsey number:

Almost Red clique size $k+1$ or Blue clique of size n.

Another Ramsey function: Let $r_k(k+1, t; n)$ be the minimum N, such that for every red/blue coloring of the k-tuples of $\{1, 2, \ldots, N\}$,



1. $k+1$ integers which induces at least t red k-tuples, or
2. n integers for which every k-tuple is blue.

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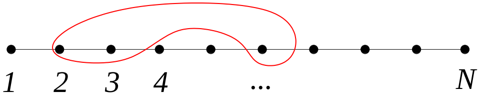

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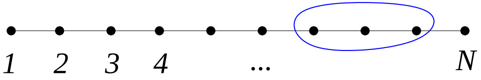

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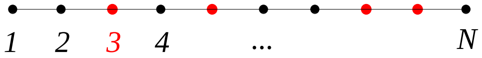

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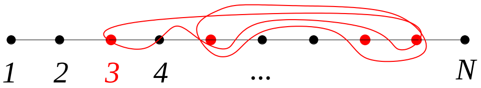

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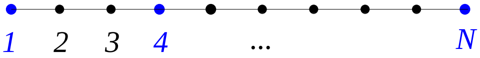

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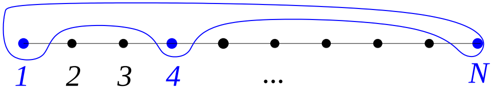

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An old problem of Erdős and Hajnal 1972

Problem (Erdős-Hajnal 1972)

For $k \geq 3$ and $t \in [k+1]$, estimate $r_k(k+1, t; n)$.

$$r_k(k+1, 1; n) = n$$

$$\vdots$$

$$k-2\left\{ 2^{\cdot^{\cdot^{\cdot^{2^{(n^{c \log n})}}}}} \right. < r_k(k+1, k+1; n) = r_k(k+1, n) < \left. 2^{\cdot^{\cdot^{\cdot^{2^{(n^{c})}}}}} \right\} k-1$$

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$$r_k(k+1, 2; n) \leq O(n^{k-1})$$

$$\vdots$$

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Upper bounds

Erdős-Rado argument: $r_k(k+1, t; n) \leq 2^{(r_{k-1}(k,t-1;n))^{k-1}}$.

Theorem (Erdős-Hajnal 1972)

$$r_k(k+1, 2; n) < cn^{k-1}$$

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Lower bounds

Erdős-Rado argument: $r_k(k+1, t; n) \leq 2^{(r_{k-1}(k, t-1; n))^{k-1}}$.

Theorem

$$c' \frac{n^{k-1}}{\log n} < r_k(k+1, 2; n) < c \frac{n^{k-1}}{\log n}$$

$$r_k(k+1, 3; n) < 2^{n^c}$$

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Theorem

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Bounds for $r_k(k+1, t; n)$

Theorem (Mubayi-S. 2018)

For $k \geq 6$, *and* $3 \leq t \leq k-2$, *we have*

$$t-1\left\{ 2^{\cdot^{\cdot^{2^{(cn^{k-t+1})}}}} \right. < r_k(k+1, t; n) < \left. 2^{\cdot^{\cdot^{2^{(c'n^{k-t+1}\log n)}}}} \right\} t-1$$

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when $k-t$ is even, and

$$t-1\left\{ 2^{\cdot^{\cdot^{\cdot^{2^{(cn^{(k-t+1)/2})}}}}} \right. < r_k(k+1, t; n) < \left. 2^{\cdot^{\cdot^{\cdot^{2^{(c'n^{k-t+1}\log n)}}}}} \right\} t-1$$

when $k-t$ is odd.

Hard cases

$$r_k(k+1, k-1; n) \leq r_k(k+1, k; n) \leq r_k(k+1, k+1; n)$$

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$$k-3\left\{ 2^{\cdot^{\cdot^{\cdot^{2^{(cn^3)}}}}} \right. < r_k(k+1, k-1; n) < \left. 2^{\cdot^{\cdot^{\cdot^{2^{(c' n^2 \log n)}}}}} \right\} k-2$$

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Key Case

Theorem (Mubayi-S. 2018)

$$2^{c'n} < r_5(6, 4; n) < 2^{2^{cn^2 \log n}}$$

$$2^{c'n^3} < r_5(6, 5; n) < 2^{2^{2^{n^c}}}$$

New Lower Bound

Theorem (Mubayi-S.-Zhu 2022)

$$2^{2^{c' n^{1/4}}} < r_5(6, 4; n) < 2^{2^{cn^2 \log n}}$$

$$2^{c' n} < r_4(5, 4; n) < 2^{2^{cn \log n}}$$

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$$k-2\left\{ 2^{\cdot^{\cdot^{\cdot^{2^{(cn)}}}}} \right. < r_k(k+1, k; n) < \left. 2^{\cdot^{\cdot^{\cdot^{2^{(c'n \log n)}}}}} \right\} k-1$$

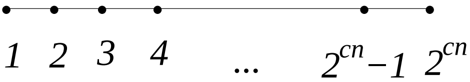
New lower bounds

Theorem (Mubayi-S.-Zhu 2022)

$$r_5(6, 4; n) > 2^{2^{cn^{1/4}}}.$$

Proof idea.

Random red/blue coloring of the pairs of $[2^{cn}]$.



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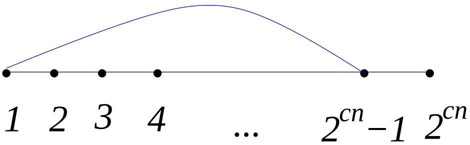
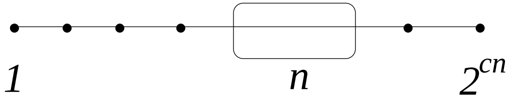


Sketch Proof

Every set of n vertices contains this pattern



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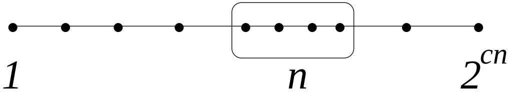

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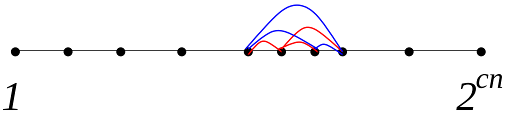

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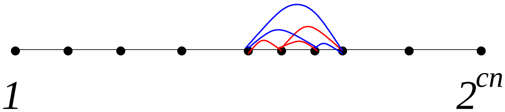
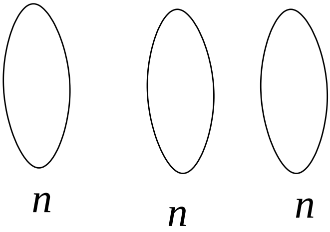


$$\binom{2^{cn}}{n}\left(\frac{63}{64}\right)^{c'n^2} < 1/3.$$

Sketch Proof

Forbidden bipartite structure:



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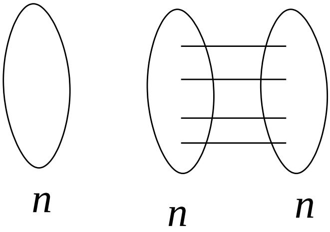

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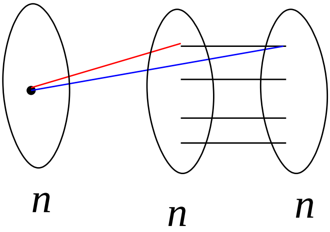

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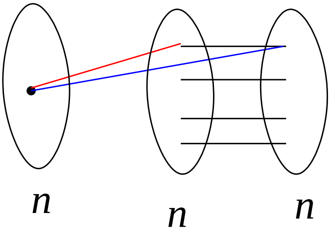
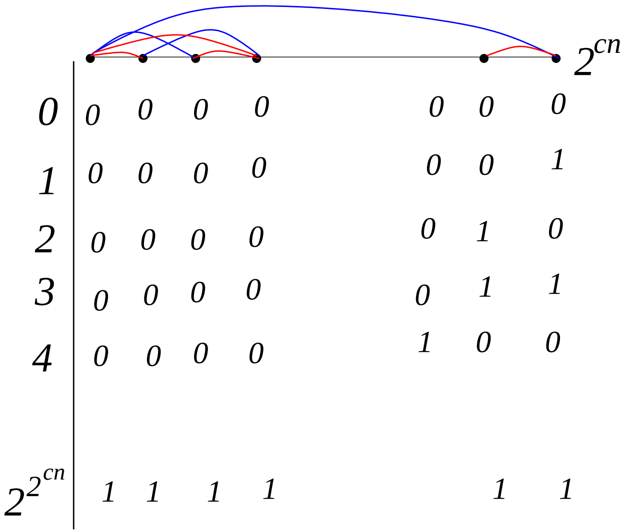


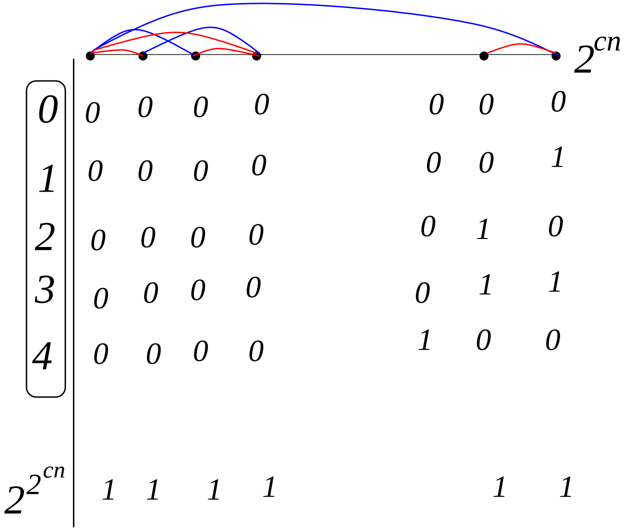
$$\binom{2^{cn}}{n}^3 n!(3/4)^{n^2} < 1/3$$

Stepping up



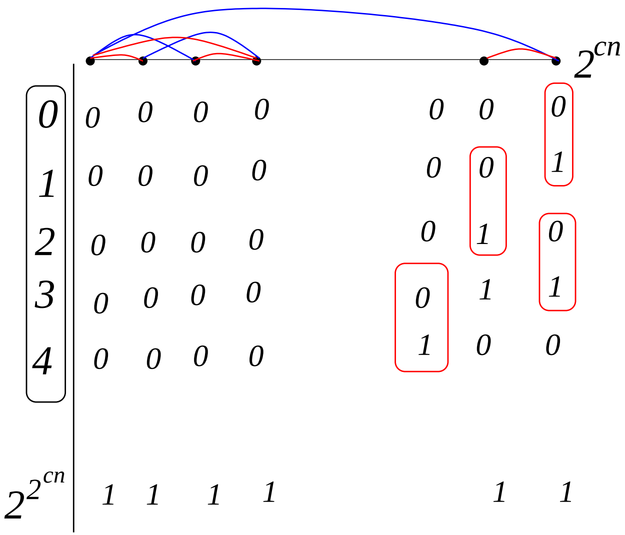
2^{cn}
0
1
2
3
4
$2^{2^{cn}}$
0 0 0 0 0 0 0
0 0 0 0 0 0 1
0 0 0 0 0 1 0
0 0 0 0 0 1 1
0 0 0 0 1 0 0
1 1 1 1 1 1

Stepping up



2^{cn}
0 | 0 0 0 0 0 0 0
1 | 0 0 0 0 0 0 1
2 | 0 0 0 0 0 1 0
3 | 0 0 0 0 0 1 1
4 | 0 0 0 0 1 0 0
$2^{2^{cn}}$ | 1 1 1 1 1 1

Stepping up



Argument

5-uniform hypergraph on $2^{2^{cn}}$ vertices.

1. Every 6 vertices induces at most 3 red edges.
2. Largest blue clique has size $O(n^4)$.

$$r_5(6, 4; n) > 2^{2^{cn^{1/4}}}.$$

Open problems

Diagonal Ramsey numbers:

$$2^{cn^2} < r_3(n,n) < 2^{2^{n^{c'}}}.$$

Off-diagonal Ramsey numbers:

$$2^{n^{c\log n}} < r_4(5,n) < 2^{2^{c'n}}.$$

Last open case for the Erdős-Hajnal problem.

$$2^{cn} < r_4(5,4;n) < 2^{2^{c'n\log n}}.$$

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Thank you!