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A Turán-Type Theorem on Chords of a Convex Polygon

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The maximum number of straight line segments connecting n points in convex position in the plane, so that no $k+1$ of them are pairwise crossing is $\binom{n}{2}$ if $n \leqslant 2k+1$ and $2kn - \binom{2k+1}{2}$ if $n \geqslant 2k+1$.

1. INTRODUCTION

One of the classical results in graph theory is Turán's theorem (see [T, B]), according to which the maximum number of edges of a graph with n vertices containing no complete subgraph on $k+1$ vertices is $\frac{1}{2}((k-1)/k)\ (n^2 - r^2) + \binom{r}{2}$, where r is the remainder of n upon division by k.

As far as we know, Paul Erdős was the first to suggest that similar questions can be raised for *geometric graphs*, i.e., for graphs whose vertices are embedded in the plane and whose edges are straight line segments. In particular, he asked the following question. What is the maximum number of

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edges of a geometric graph with n vertices in general position, if it does not contain $k+1$ pairwise disjoint edges? For $k=1$ this maximum is equal to n, as was proved by Erika Pannwitz and H. Hopf (see [E]) and by Yaakov S. Kupitz [K1]. For $k=2$, Noga Alon and Paul Erdős established the upper bound $6(n-1)$, which has been recently improved by Paul M. O'Donnel and Micha A. Perles [OP] to about $3.6n$. The lower bound $\frac{5}{2}(n-1)$ is due to Perles. The case $k>2$ is completely open.

Of course, the same question can be asked for geometric graphs whose n vertices form a convex polygon. As Kupitz observed, if such a graph does not contain $k+1$ pairwise disjoint edges, then it cannot have more than nk edges, and this bound is tight.

In this note we consider the "dual" problem. What is the maximum number of edges of a geometric graph whose n vertices are in convex position and which does not contain $k+1$ pairwise crossing edges? We have the following result:

THEOREM. *Let $f_k(n)$ denote the maximum number of straight line segments connecting n points in convex position in the plane so that no $k+1$ of them are pairwise crossing. Then*

$$f_k(n)=\begin{cases}\binom{n}{2} & \text{if } n\leqslant 2k+1\\ 2kn-\binom{2k+1}{2} & \text{if } n\geqslant 2k+1.\end{cases}$$

This question was raised by Bernd Gärtner. The above theorem was conjectured by Micha A. Perles, and for $k=2$ it was also proved independently by Imre Ruzsa. The weaker bound $f_k(n)<k^3 2^k n$ follows from a theorem of András Gyárfás [Gy], which estimates the chromatic number of the intersection graph associated with a system of chords of a convex n-gon. There are no nontrivial results concerning this problem if we drop the condition that the points are in convex position.

Many related problems and results on geometric graphs can be found in [K2, GyL, Tu, MP].

2. PROOF OF THE THEOREM

Let $k\geqslant 1$ be fixed. For $n\leqslant 2k+1$ the statement is trivial. Assume that we have already proved it for every natural number smaller than n, and consider a graph G consisting of a maximum number $(f_k(n))$ of straight line segments connecting the vertices of a convex n-gon such that no $k+1$ cross each other.

Claim 1. If two vertices are separated by fewer than k points along the boundary of our convex n-gon, then they are connected by an edge of G.

If there is no edge ab whose endpoints are separated by at least k vertices along (both arcs of) the boundary of the n-gon, then the number of edges $e(G) \leqslant kn$. So assume that there is such an edge ab, and let $p_1, p_2, ..., p_{n_1}$; $q_1, q_2, ..., q_{n_2}$ denote the vertices of the polygon on the right-hand side and on the left-hand side of ab, respectively, listed in clockwise order. (See Fig. 1.)

Obviously, $n_1 \geqslant k$, $n_2 \geqslant k$.

We define a partial order "$\prec$" on the set of all edges of G that cross ab, as follows. Two edges $q_i p_j$ and $q_{i'} p_{j'}$ are comparable if and only if they cross each other, and

$$q_i p_j \prec q_{i'} p_{j'} \Leftrightarrow i < i' \text{ and } j < j'.$$

This is clearly a transitive relation.

We say that the minimal elements in this partial order have rank 1. In general, we call the minimal elements of the partial order obtained by the deletion of all elements of rank smaller than r, *edges of rank* r. In other words, $\text{rank}(q_i p_j)$ is defined as the largest integer r such that there is a sequence of edges

$$q_{i_1} p_{j_1} \prec q_{i_2} p_{j_2} \prec \cdots \prec q_{i_r} p_{j_r} = q_i p_j.$$

In particular, it follows that

1. no two edges of the same rank cross each other,
2. the rank of every edge is at most $k-1$ (otherwise, $q_{i_1} p_{j_1}, ..., q_{i_k} p_{j_k}$, together with ab would form a system of $k+1$ pairwise crossing edges).

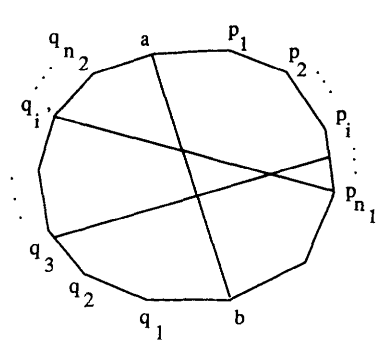

FIGURE 1

Let us define a convex geometric graph G_1 with n_2+k+1 vertices $\{a, p_1^*, p_2^*, ..., p_{k-1}^*, b, q_1, q_2, ..., q_{n_2}\}$ (in clockwise order), as follows.

Let G_1 be the same as G, when restricted to $\{a, b, q_1, q_2, ..., q_{n_2}\}$. Let $q_i p_r^*$ be in $E(G_1)$ if and only if there is an edge $q_i p_j \in E(G)$ whose rank is r. (Note that this edge is not necessarily uniquely determined.) Finally, let $\{a, p_1^*, p_2^*, ..., p_{k-1}^*, b\}$ induce a subgraph in G_1 consisting of the single edge ab.

CLAIM 2. *G_1 does not have $k+1$ pairwise crossing edges.*

Proof. It is sufficient to show that if there are t pairwise crossing edges $q_{i_1} p_{r_1}^*, q_{i_2} p_{r_2}^*, ..., q_{i_t} p_{r_t}^*$ in G_1, $i_1 < i_2 < \cdots < i_t$, $r_1 < r_2 < \cdots < r_t$, then one can find t pairwise crossing edges in G, all of which cross ab and whose left endpoints are in the interval $\{q_i : i_1 \leqslant i \leqslant i_t\}$.

Pick t edges $q_{i_1} p_{j_1}, q_{i_2} p_{j_2}, ..., q_{i_t} p_{j_t} \in E(G)$ with $\operatorname{rank}(q_{i_1} p_{j_1}) = r_1$, $\operatorname{rank}(q_{i_2} p_{j_2}) = r_2, ..., \operatorname{rank}(q_{i_t} p_{j_t}) = r_t$.

We shall produce a sequence $q_{u_\mu} p_{v_\mu}$ $(\mu = 1, ..., t)$ of pairwise crossing edges of G, such that:

1. $\operatorname{rank}(q_{u_\mu} p_{v_\mu}) = r_\mu$
2. $u_\mu \geqslant i_\mu$ for $\mu = 1, ..., t$.

We proceed by reverse induction on μ. First define $q_{u_t} p_{v_t} = q_{i_t} p_{j_t}$. Assume we have already chosen $q_{u_\mu} p_{v_\mu}$ for $\mu = t, t-1, ..., s$, $s > 1$, so as to satisfy the requirements above. There is an edge $q_u p_v$ of rank r_{s-1}, that crosses $q_{u_s} p_{v_s}$. If $u \geqslant i_{s-1}$, choose this edge to be $q_{u_{s-1}} p_{v_{s-1}}$, and we are done. If not, then $q_u p_v$ lies "below" $q_{i_{s-1}} p_{j_{s-1}}$ (see Fig. 2), since two edges of the same rank do

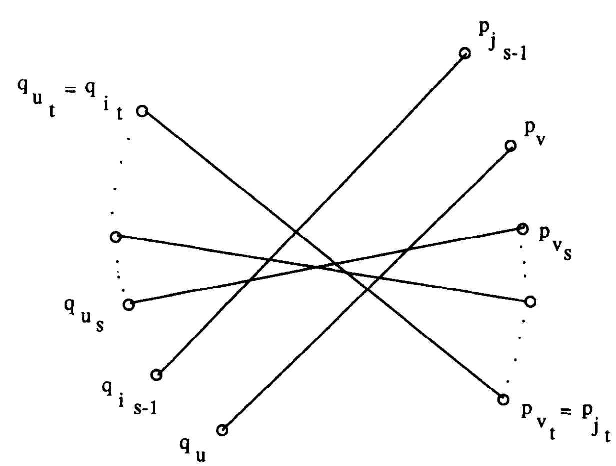

FIGURE 2

not cross. It follows that $q_{i_{s-1}}p_{j_{s-1}}$ must cross $q_{u_s}p_{v_s}$, and hence all the edges $q_{u_\mu}p_{v_\mu}$, $s \leqslant \mu \leqslant t$. In this case choose $q_{i_{s-1}}p_{j_{s-1}}$ to be $q_{u_{s-1}}p_{v_{s-1}}$. ∎

CLAIM 3. *The number of edges of* G_1, $e(G_1) \leqslant f_k(n_2+k+1)-k^2+k$.

Proof. It is sufficient to show that we can add k^2-k edges to G_1 without creating $k+1$ pairwise intersecting edges. In view of Claim 1, all edges of G_1 whose endpoints are separated by fewer than k points and which are not in G_1, can be added to it.

Observe that, for every edge $q_i p_j \in E(G)$, $1 \leqslant i \leqslant k-1$,

$$\operatorname{rank}(q_i p_j) \leqslant i$$

holds. This yields that

$$q_i p_r^* \notin E(G_1) \text{ whenever } r > i.$$

Furthermore, all edges within $\{a, p_1^*, p_2^*, \ldots, p_{k-1}^*, b\}$ except ab are missing from G_1. Hence, at least

$$\sum_{i=1}^{k-1} (k-i-1) + \binom{k+1}{2} - 1 = k^2 - k$$

edges can be added to G_1 without creating $k+1$ pairwise crossing edges. ∎

Similarly, we can define a graph G_2 with n_1+k+1 vertices $\{a, p_1, p_2, \ldots, p_{n_1}, q_1^*, q_2^*, \ldots, q_{k-1}^*\}$, by connecting q_r^* and p_j by an edge of G_2 if and only if there exists an edge $q_i p_j \in E(G)$ with $\operatorname{rank}(q_i p_j) = r$, and letting $\{a, p_1, p_2, \ldots, p_{n_1}, b\}$ induce the same subgraph in G_2 as in G. Just like before, $e(G_2) \leqslant f_k(n_1+k+1)-k^2+k$.

Let $\deg_{G_1}(p_r^*)$ (and $\deg_{G_2}(q_r^*)$) be the number of points in G_1 (resp. G_2) adjacent to p_r^* (resp. q_r^*).

CLAIM 4. $\deg_{G_1}(p_r^*) + \deg_{G_2}(q_r^*) - 1$ *is at least as large as* e_r, *the number of edges of* G *having rank* r $(1 \leqslant r \leqslant k-1)$.

Proof. This follows immediately from the fact that the edges of rank r form a forest on $\deg_{G_1}(p_r^*) + \deg_{G_2}(q_r^*)$ vertices. ∎

Since $ab \in E(G_1) \cap E(G_2)$, we obtain by Claim 4

$$\begin{aligned} e(G) &= e(G_1) + e(G_2) - 1 - \sum_{r=1}^{k-1} \{\deg_{G_1}(p_r^*) + \deg_{G_2}(q_r^*) - e_r\} \\ &\leqslant e(G_1) + e(G_2) - k. \end{aligned}$$

This, combined with Claim 3, implies that

$$e(G) \leqslant f_k(n_1 + k + 1) + f_k(n_2 + k + 1) - 2k^2 + k.$$

According to our assumptions, $n_1 + n_2 + 2 = n$, $n_1 \geqslant k$, $n_2 \geqslant k$. Therefore,

$$2k + 1 \leqslant n_i + k + 1 < n \qquad (i = 1, 2),$$

and we can apply the induction hypothesis to obtain

$$\begin{aligned} e(G) &\leqslant 2k(n_1 + k + 1) + 2k(n_2 + k + 1) - 2\binom{2k+1}{2} - 2k^2 + k \\ &= 2k(n_1 + n_2 + 2) - 2k^2 - k \\ &= 2kn - \binom{2k+1}{2}, \end{aligned}$$

as desired.

This estimate is tight, as is shown by the following construction: Let $p_1, p_2, ..., p_n$ be the vertices of a convex n-gon in clockwise order, and let p_i and p_j $(i < j)$ be joined by an edge if and only if

1. $i \leqslant k$, or

2. p_i and p_j are separated by fewer than k points on (the shorter arc of) the boundary of the n-gon.

(See Fig. 3.)

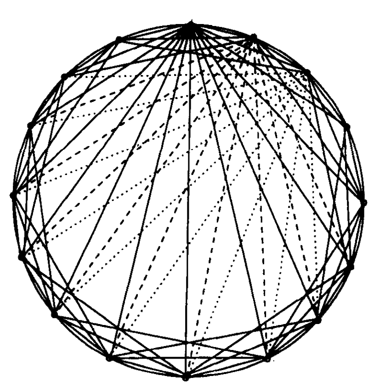

FIG. 3. The lower bound construction for $n = 16$ and $k = 3$. There are 75 chords and this is optimal.

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