Edge intersection graphs
of systems of grid paths
with bounded number of bends

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Abstract

We answer some of the questions raised by Golumbic, Lipshteyn and Stern regarding edge intersection graphs of paths on a grid (EPG graphs). We prove that for any $d \geq 4$, in order to represent all $n$ vertex graphs with maximum degree $d$ as edge intersection graphs of $n$ paths, a grid of area $\Theta(n^2)$ is needed. A bend is a turn of a path at a grid point. Let $B_k$ be the class of graphs that have an EPG representation such that each path has at most $k$ bends. We show several results related to the classes $B_k$; among them we prove that for any odd integer $k$, $B_k \subsetneq B_{k+1}$. Lastly, we show that only a very small fraction of all the $2^{\binom{n}{2}}$ labeled graphs on $n$ vertices is in $B_k$.

Key words: Edge intersection graph; graph drawing.

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1 Introduction

In a recent paper [5], Golumbic, Lipshteyn and Stern introduced a notion of edge intersection graph of paths on a grid (EPG graphs) and studied some of their properties. Their research was motivated by studying graphs that come from circuit layout on a grid; see [5] for details, and [3,9] for more circuit layout problems.

Consider a family \( \mathcal{F} \) of paths on a grid. The edge intersection graph of \( \mathcal{F} \) is the graph whose vertices correspond to the paths from \( \mathcal{F} \), two vertices being connected if and only if the corresponding paths share an edge of the grid (see Fig. 1 for an example). An edge intersection graph of paths on a grid (an EPG graph) is a graph which may be represented in this way. Thus EPG graphs generalize edge intersection graphs of paths on a tree (more specifically, on a tree of degree 4, studied in [6]).

![Fig. 1. An EPG realization of a graph](image)

A bend is a turn of a path at a grid point. Denote by \( B_k \) the class of graphs that have an EPG representation such that each path has at most \( k \) bends. Golumbic et al. [5] studied these families, especially \( B_1 \). This research was motivated by the fact that in chip manufacturing, bends result in increasing the costs. In particular, they showed:

**Theorem 1** *(Golumbic, Lipshteyn and Stern [5]:)*
Every graph with \( n \) vertices has an EPG representation on an \( n \times 2n \) grid.

For each \( k \), \( B_k \subseteq B_{k+1} \); strong inclusions are conjectured.

Each tree is \( B_1 \). (Follows \( B_0 \subseteq B_1 \) because not every tree is an interval graph.)

\( K_{3,3} \) and \( K_{3,3} \setminus \{e\} \) (i.e. \( K_{3,3} \) with one edge deleted) are not \( B_1 \).

For any \( n \), the complete bipartite graph \( K_{m,n} \) is \( B_{2m-2} \), see Fig. 2.

In the present paper, we shall refine some of these results and answer some questions from [5].

Let \( f_d(n) \) denote the minimum grid area required to represent all graphs on \( n \) vertices with maximum degree \( d \). Recall (see Theorem 1(1)) that all \( n \) vertex graphs can be represented in an \( n \times 2n \) grid. Since \( K_{n/2,n/2} \) is a triangle free graph with \( n^2/4 \) edges, we need a grid of area \( \Omega(n^2) \) in order to represent it as an EPG. Hence \( f_{n-1}(n) = \Theta(n^2) \). However, in Section 2 we shall show that even for small \( d \), a grid of size \( \Theta(n^2) \) is required:

**Theorem 2** For fixed \( d \geq 4 \) and sufficiently large \( n \), \( f_d(n) = \Theta(n^2) \).

In Section 3 we show several results concerning the classes \( B_k \):

**Theorem 3**

1. \( K_{2,m} \) is \( B_1 \) if and only if \( m \leq 4 \). (In particular, \( K_{2,5} \) is not \( B_1 \).)
(2) For fixed \( m \) and sufficiently large \( n \), \( K_{m,n} \) is not \( B_{2m-3} \) (for shortness, we write this as “\( K_{m,\infty} \) is not \( B_{2m-3} \).”).

(3) For an odd \( k \), we have \( B_k \subsetneq B_{k+1} \).

(4) \( K_{m,n} \) is \( B_{\max\{\lceil m/2\rceil,\lceil n/2\rceil\}} \). In particular, \( K_{m,m} \) is \( B_{\lceil m/2 \rceil} \).

In Section 4 we prove that only a very small fraction of all the \( 2^{(\binom{n}{2})} \) labeled graphs on \( n \) vertices is in \( B_k \):

**Theorem 4** The number of labeled graphs on \( n \) vertices which can be represented as \( B_k \)-EPG is \( 2^{O(kn \log(kn))} \).

2 The area of the grid

In this section, we will prove Theorem 2. Clearly \( f_d(n) \leq 4n^2 \), since all \( n \) vertex graphs can be represented in an \( n \times 2n \) grid. For the lower bound, we will use a triangle free expander graph.

*Observation:* For fixed \( d \geq 4 \) and sufficiently large \( n \), there exists a triangle free graph \( G \) with \( n \) vertices and maximum degree at most \( d \), such that for every \( S \subset V(G) \) with \( |S| \leq n/2 \), we have \( \Gamma(S) \geq (2/3)|S| - (d - 1)^3/3 \) (we denote by \( \Gamma(S) \) the set of vertices in \( V(G) \setminus S \) that have neighbors in \( S \)).

*Proof.* Assume \( d \geq 4 \) and is even. If \( d \) is odd, then just set \( d \leftarrow d - 1 \). Now generate a random \( d \)-regular graph on \( n \) vertices \( G \) by taking \( d/2 \) permutations of \( V = \{1, 2, ..., n\} \), \( \pi_1, ..., \pi_{d/2} \), with each \( \pi_i \) chosen uniformly among all \( n! \) permutations and with all \( \pi_i \) independent. Then we have
\[ E(G) = \{(i, \pi_j(i)), (i, \pi_j^{-1}(i)) : j = 1, \ldots, d/2, \ i = 1, \ldots, n\} \]

Now the eigenvalues of the adjacency matrix of an undirected graph \( G \) are real and can be ordered

\[ \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \]

Since \( G \) is \( d \)-regular, \( \lambda_1 = d \). Friedman [7] showed that for fixed \( d \) and sufficiently large \( n \), \( \lambda_2(G) > 2\sqrt{d-1} + .1 \) holds with probability less than 1/10.

Now let \( X \) denote the number of triangles in \( G \). Bollobas [2] showed that

\[ \mathbb{E}[X] = \frac{(d-1)^3}{6}(1 + o(1)) \]

Using Markov Inequality, we obtain

\[ \mathbb{P}[(\lambda_2(G) > 2\sqrt{d-1} + .1) \cup (X > (d-1)^3/3)] \leq \]

\[ \leq \mathbb{P}[\lambda_2(G) > 2\sqrt{d-1} + .1] + \mathbb{P}[X > (d-1)^3/3] \leq \frac{1}{10} + \frac{\mathbb{E}[X]}{(d-1)^3/3} \leq \frac{1}{10} + \frac{1}{10} \leq 1. \]

Hence there exists a \( d \)-regular graph with \( n \) vertices such that \( \lambda_2(G) \leq 2\sqrt{d-1} + .1 \), and the number of triangles in \( G \) is less than \( (d-1)^3/3 \). By Alon and Milman [1], for every \( S \subset V(G) \) such that \( |S| \leq n/2 \), we have

\[ \Gamma(S) \geq \frac{2(d - (2\sqrt{d-1} + .1))}{3d - 2(2\sqrt{d-1} + .1)} |S| \geq \frac{1}{3} |S|. \]
Now we can delete at most \((d-1)^3/3\) edges in order to remove all the triangles in \(G\), and then remove all multiple edges and loops. Hence we have a triangle free graph on \(n\) vertices and maximum degree \(d\), such that for every \(S \subset V(G)\) such that \(|S| \leq n/2\), \(\Gamma(S) \geq (1/3)|S| - (d - 1)^3/3\).

\[\square\]

**Proof of Theorem 2.** Let \(G\) be the \(n\) vertex triangle free expander graph described in the observation above. Then let \(\mathcal{P}\) be a collection of \(n\) paths in an \(s \times t\) grid such that the edge intersection graph of \(\mathcal{P}\) is \(G\). We claim that there exists a vertical line \(l\) which intersects at least \(n/10\) paths. Indeed, assume there is no such line, and let \(R(l)\) denote the paths which lay completely right to \(l\) and not intersecting \(l\), and denote by \(I(l)\) the set of paths intersecting \(l\).

By assumption, we can move the line \(l\) such that \((1/2 - 1/10)n < |R(l)| \leq n/2\). By construction of \(G\), this implies for sufficiently large \(n\)

\[|I(l)| > (1/3)(1/2 - 1/10)n - \frac{(d - 1)^3}{3} \geq n/10.\]

Hence we have a contradiction. Since \(G\) is triangle free, this implies

\[2t \geq |I(l)| \geq n/10 \quad \Rightarrow \quad t \geq n/20.\]

By the same argument, \(s \geq n/20\), which implies

\[\frac{n^2}{400} \leq f_d(n).\]

\[\square\]
Proof of Theorem 3(1). Consider a realization of $K_{2,m}$ as $B_1$. Denote the paths corresponding to the “left” side of $K_{2,m}$ by $a_1$, $a_2$, those corresponding to the “right” side by $b_1, b_2, \ldots, b_m$. Denote the point of bend of $a_i$ by $A_i$. Consider the following cases:

(1) $A_1$ and $A_2$ coincide;

(2) $A_1$ and $A_2$ do not coincide but lay on the same horizontal or vertical line;

(3) $A_1$ and $A_2$ do not lay on the same horizontal or vertical line.

In Case 1, each $b_i$ passes through $A = A_1 = A_2$ and contains points of a segment of each $a_i$ in the neighborhood of $A$. Therefore it is possible to add at most two $b_i$s so that they do not meet.

In Case 2, each $b_i$ has a horizontal segment passing through $A$, the midpoint of $A_1A_2$. Therefore it is possible to add at most one $b_i$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{Cases 1 and 2 in the proof of Theorem 3(1)}
\end{figure}

In Case 3, denote by $D_1$ the point of intersection of the horizontal line containing $A_1$ and the vertical line containing $A_2$, and denote by $D_2$ the point of intersection of the vertical line containing $A_1$ and the horizontal line containing $A_2$. The paths $a_1$ and $a_2$ may form several configurations – three of them appear on Fig. 4. However, it is easy to see that in all cases each $b_i$ has the
bend at $D_1$ or at $D_2$. Therefore it is possible to add at most four $b_i$s: at most two having the bend at $D_1$, and at most two having the bend at $D_2$.

![Diagram](image)

Fig. 4. Case 3 in the proof of Theorem 3(1)

Therefore for $m \geq 5$, the graph $K_{2,m}$ is not $B_1$. A realization of $K_{2,4}$ as $B_1$ is shown at Fig. 5.

![Diagram](image)

Fig. 5. $K_{2,4}$ as $B_1$

**Proof of Theorem 3(2) and 3(3).** Let $m$ be fixed, and let $n$ be a very large number. Consider a [hypothetic] realization of $K_{m,n}$ as $B_{2m-3}$. Let $a_1, a_2, \ldots, a_m$ be the paths corresponding to the “left” side of $K_{m,n}$, $b_1, b_2, \ldots, b_n$ the paths corresponding to its “right” side.

Each $a_i$ consists of $2m-2$ segments. For each $b_j$ consider the set of $m$ segments, one segment from each $a_i$, that meet $b_j$ (if $b_j$ meets several segments of an $a_i$, choose one of them). There are $(2m-2)^m$ sets that may be obtained in this
way: one of $2m - 2$ segments is chosen for each of $m$ $a_i$s. Therefore, there is a set $A = \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m\}$, each $\tilde{a}_i$ being a segment of $a_i$, and $B$, a subset of \{${b_1, b_2, \ldots, b_n}$\} of size at least $\frac{n}{(2m-2)^m}$, such that all the members of $B$ (paths) meet all the members of $A$ (segments). For each $b_j \in B$, denote by $\tilde{b}_{ij}$ the segment of $b_j$ that meets $\tilde{a}_i$ (if some $b_j$ has several such segments, choose one of them).

Let $L$ be the set of all vertical and horizontal lines that contain $\tilde{a}_i$s. There are at most $m$ lines in this set. Let $P$ be the set of points of intersection of lines from $L$. The set $P$ contains at most $(m/2)^2$ points (on the other hand it may be empty – if all the lines in $L$ are parallel).

Delete from $B$ all $b_j$s that have a point from $P$ as an inner point (there are at most $2(m/2)^2$ such paths).

Delete from $B$ also all such $b_j$s that $\tilde{b}_{ij}$ is not included in the interior of $\tilde{a}_i$ for some $i$. (That is, we delete $b_j$s which have a $\tilde{b}_{ij}$ with at most one endpoint inside $\tilde{a}_i$.) At most $2m$ members of $B$ are deleted at this stage. Denote the obtained subset of $B$ by $C$. Since $n$ was assumed very large, $C$ is not empty.

Consider a path $b_j$ in $C$. For each $1 \leq i \leq m$, $b_j$ has a segment $\tilde{b}_{ij}$ contained in an $\tilde{a}_i$. Both endpoints of $\tilde{b}_{ij}$ belong to a unique $\tilde{a}_i$. Therefore $b_j$ has at least $2m - 2$ bends.

In particular, it follows from this result that $2m - 2$ in Theorem 1(5) is tight: $B_{m,\infty}$ is $B_{2m-2}$ but not $B_{2m-3}$. This proves Theorem 3(3): $B_k \subseteq B_{k+1}$ for an odd $k$. This also means that there is no natural $k$ such that each graph is $B_k$. 9
Proof of Theorem 3(4). We start with a construction that shows that $K_{m,n}$ is $B_{\max\{m,n\}-1}$. Fig. 6 presents it for $K_{6,8}$. It is easy to see that it also works when $m$ or $n$ is odd: If we delete, say, the rightmost “vertical” path, then each “horizontal” path has an unnecessary bend and the “last” segment can be deleted.

![Image of Fig. 6](image)

Fig. 6. $K_{6,8}$ as $B_7$ – an example of $K_{m,n}$ as $B_{\max\{m,n\}-1}$

However, this construction may be improved. The following construction is a realization of $K_{m,n}$ as $B_{\max\{\lceil m/2 \rceil,\lceil n/2 \rceil\}}$. Fig. 7 presents it for $K_{6,8}$; it is clear how to generalize it for any even $m$ and $n$; if $m$ or/and $n$ is odd, use the construction with $m + 1$ or/and $n + 1$ and delete the extra path(s). □

We conjecture that this construction is the best possible. In other words, that $K_{m,n}$ is not $B_{\max\{\lceil m/2 \rceil,\lceil n/2 \rceil\}-1}$. In particular, that $K_{m,m}$ is not $B_{\lceil m/2 \rceil-1}$. If this conjecture is true, it would prove the strong inclusion $B_k \subsetneq B_{k+1}$ for each $k$. 

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4 Bounding the number of $n$ vertex graphs in $B_k$

Proof of Theorem 4. Since each path has at most $k$ bends, then for $r = \lceil (k/2) + 1 \rceil$, each path consists of at most $r$ horizontal and $r$ vertical segments. Each path $p_i$ can be described by the relations

\[
\begin{align*}
  x + y_{i,1}, & \quad c_{i,1} \leq x \leq d_{i,1} & \quad x_{i,1} + y, & \quad c_{i,r+1} \leq y \leq d_{i,r+1} \\
  x + y_{i,2}, & \quad c_{i,2} \leq x \leq d_{i,2} & \quad x_{i,2} + y, & \quad c_{i,r+2} \leq y \leq d_{i,r+2} \\
  \vdots & & \vdots & \vdots \\
  x + y_{i,r}, & \quad c_{i,r} \leq x \leq d_{i,r} & \quad x_{i,2r} + y, & \quad c_{i,2r} \leq y \leq d_{i,2r}
\end{align*}
\]
We will assume that there are exactly \( r \) equations that represents the horizontal segments of \( p_i \) and \( r \) equations that represents the vertical lines of \( p_i \).

If there are fewer than \( r \) horizontal (or vertical) segments, we can just repeat any of the relations. Whether paths \( p_i \) and \( p_j \) cross, depends on the signs of the polynomials

\[
P_{i,j,s,t} = x_{i,s} - x_{j,t} \quad Q_{i,j,s,t} = y_{i,s} - y_{j,t} \quad R_{i,j,s,t} = d_{i,s} - c_{j,t} \quad S_{i,j,s,t} = d_{j,t} - c_{i,s}
\]

for \( s, t \in \{1, 2, ..., r\} \), and \( i, j \in \{1, 2, ..., n\} \) with \( i \neq j \). Hence we have \( 2r^2 \binom{n}{2} \) polynomials over \( 6rn \) variables with degree 1. Now recall that the Milnor-Thom theorem says that the number of sign patterns for \( m \) polynomials of degree \( d \) in \( v \) variables is at most \((4edm/v)^v \) \( [8,10,11] \). Hence the number of sign patterns over our \( 2r^2 \binom{n}{2} \) polynomials over \( 6rn \) variables with degree 1 is at most

\[
\left( \frac{4e2r^2n^2}{6kn} \right) \leq 2^{O(kn \log kn)}
\]

Hence the number of edge-intersection relationships that can be defined on \( n \) \( k \)-bend-paths in a grid is \( 2^{O(kn \log kn)} \).

\[\square\]

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