A note on the clique chromatic number of geometric graphs

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Abstract

Given a finite point set V in $\mathbb{R}^d$ and a parameter $r > 0$, the corresponding *geometric graph G in* $\mathbb{R}^d$ is the graph whose vertex set is $V \subset \mathbb{R}^d$, and vertices $u, v \in V$ are adjacent in G if and only if $||u - v|| \leq r$. In this note, we study the *clique chromatic number* of geometric graphs, that is, the minimum number of colors required to color the vertices of a graph such that every maximal clique of size at least two is not monochromatic. Improving the results of McDiarmid, Mitsche, and Pralat, we show that if G is a geometric graph in $\mathbb{R}^d$ then $\chi_c(G) \leq 2^{O(d)}$. From the other direction, we show that there are geometric graphs G in $\mathbb{R}^d$ such that $\chi_c(G) > \Omega(d^{1/2}(\log d)^{-1/2})$.

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1 Introduction

Given a simple graph $G = (V,E)$, a *proper coloring* of G is a coloring of the vertices such that no two adjacent vertices have the same color. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors required to properly color G. A *clique coloring* of G is a coloring of the vertices such that no maximal clique of size at least two is monochromatic. The *clique chromatic number* of G, denoted by $\chi_c(G)$, is the minimum number of colors required to clique color G. Clearly we have $\chi_c(G) \leq \chi(G)$ for every graph G, and if G is triangle-free, then we have $\chi(G) = \chi_c(G)$. The clique chromatic number has been studied in many papers, and we refer the interested reader to [2, 3, 6, 5].

McDiarmid, Mitsche, and Pralat [5] initiated the study of the clique chromatic number of geometric graphs in $\mathbb{R}^d$. Given a finite point set V in $\mathbb{R}^d$ and a parameter $r > 0$, the corresponding *geometric graph G in $\mathbb{R}^d$* is the graph whose vertex set is $V \subset \mathbb{R}^d$, and vertices $u, v \in V$ are adjacent in G if and only if $||u - v|| \leq r$, where $||u - v||$ denotes the Euclidean distance between u and v. McDiarmid et al. [5] showed that if G is a geometric graph in $\mathbb{R}^d$, then $\chi_c(G) \leq 2(\sqrt{d}+1)^d$. In the other direction, they showed for each $d \geq 1$, there are geometric graphs G in $\mathbb{R}^d$ such that $\chi_c(G) > \Omega(d^{1/4}(\log d)^{-1/2})$. In what follows, we improve both of these bounds.

Theorem 1.1. *The clique chromatic number of any geometric graph G in $\mathbb{R}^d$ satisfies*

$$\chi_c(G) \leq 2^{O(d)}.$$

Theorem 1.2. *For every integer $d \geq 1$, there exists a geometric graph G_d in $\mathbb{R}^d$ whose clique chromatic number satisfies*

$$\chi_c(G_d) \geq \Omega(d^{1/2}(\log d)^{-1/2}).$$

2 A new upper bound

Proof of Theorem 1.1. Let $G = (V,E)$ be a geometric graph in $\mathbb{R}^d$ with parameter $r > 0$, where $V \subset \mathbb{R}^d$. Let B_0 be a sufficiently large ball with radius R such that $V \subset B_0$, and let $\mathscr{C} = \{B_1, \ldots, B_m\}$ be a maximum packing of B_0, where B_i is a ball centered at $v_i \in \mathbb{R}^d$ and with radius $r/4$. Note that $\mathscr{C}$ is finite by volume considerations, and in fact, $m = |\mathscr{C}| \leq (R/(r/4))^d$. For each i, let B_i' and B_i'' be balls centered at v_i with radius $r/2$ and r respectively. Set $\mathscr{C}' = \{B_1', \ldots, B_m'\}$ and $\mathscr{C}'' = \{B_1'', \ldots, B_m''\}$. Notice that for any point $x \in B_0$, there is a ball $B_i \in \mathscr{C}$ such that $||x - B_i|| < r/4$, which implies that $x \in B_i'$. In other words, $B_0 \subset \bigcup_{i=1}^m B_i'$.

Let G'' be the intersection graph of the balls in $\mathscr{C}''$, that is, $V(G'') = \mathscr{C}''$, and two elements $B_i'', B_j'' \in \mathscr{C}''$ are adjacent in G'' if and only if $B_i'' \cap B_j'' \neq \emptyset$. Notice that G'' has maximum degree at most 12^d. Indeed, for fixed i, the number of balls from $\mathscr{C}''$ that intersect B_i'' is equal to the number of balls from $\mathscr{C}''$ that lie within $B(v_i, 3r)$, where $B(v_i, 3r)$ is the ball centered at v_i with radius $3r$. This is at most the number of balls from $\mathscr{C}$ that can be packed in $B(v_i, 3r)$. Using that $\mathscr{C}$ is a packing of balls of radius $r/4$, the number of balls in $\mathscr{C}$ that lie entirely within $B(v_i, 3r)$ is at most

$$Vol(B(v_i, 3r))/Vol(B_i) = 12^d.$$

As the graph G'' has maximum degree at most 12^d, it has chromatic number $k \leq 12^d + 1$. So there is a mapping $c : [m] \to [k]$ such that if B_i'' and B_j'' have a nonempty intersection, then $c(i) \neq c(j)$. Now we color the points in V with $2k$ colors of the form (j,t) with $1 \leq j \leq k$ and $t \in \{1,2\}$, as follows. For $v \in V$, let $i(v)$ be the minimum index i such that $v \in B_i'$. Let S_i be the set of vertices $v \in V$ for which $i(v) = i$. Thus, $S_1, \ldots, S_m$ form a partition of V. If $|S_i| = 1$, then color the vertex in S_i with color $(c(i), 1)$. If $|S_i| > 1$, then color one vertex (chosen arbitrarily) in S_i with color $(c(i), 1)$, and color the rest of the vertices in S_i with color $(c(i), 2)$. Since any two points in V that are adjacent in G have distance at most r, they either belong to the same part S_i, or belong to two different parts S_i and S_j such that $B_i'' \cap B_j'' \neq \emptyset$. Let K be a maximal clique in V. If K contains two vertices u, v that belong to different parts S_i and S_j, then by the coloring c, vertices u and v also get different colors by considering the first coordinate, so K is not monochromatic. Otherwise, there is an i such that $i(v) = i$ for all vertices v in K. So, all vertices of K are in the ball B_i', which has diameter r. By maximality of K, all vertices of V in B_i' and, in particular, all vertices of S_i are in K. Since there are two vertices in S_i that receive different colors, K is not monochromatic. We have thus shown that no maximal clique of G with at least two vertices is monochromatic, so

$$\chi_c(G) \leq 2k \leq 2(12^d + 1).$$

This completes the proof. □

3 A new lower bound

For the proof of Theorem 1.2, we will need the following lemma.

Lemma 3.1 (Alon, Ben-Shimon, Krivelevich [1]). *There is a positive constant C so that for every natural number d there exists a regular triangle-free graph G on d vertices with independence number $\alpha(G) < C\sqrt{d \log d}$. Moreover, G is Δ-regular where $\Delta = \Theta(\sqrt{d \log d})$.*

Proof of Theorem 1.2. Let $d \geq 1$ and let G be a Δ-regular graph on the vertex set $V(G) = \{1, \ldots, d\}$ meeting the requirements of Lemma 3.1. Since G is triangle-free, we have

$$\chi_c(G) = \chi(G) \geq \frac{d}{\alpha(G)} \geq \Omega(\sqrt{d/\log d}).$$

Set $r = \sqrt{2\Delta - 1}$. We can assume that d is sufficiently large so that $\sqrt{2\Delta - 2} > 0$, since otherwise the statement easily follows. We now use an argument due to Frankl and Maehara [4] to construct a d-element point set $V = \{v_1, \ldots, v_d\}$ in $\mathbb{R}^d$ such that $||v_i - v_j|| \leq r$ if and only if $(i, j) \in E(G)$.

Let $A = (a_{i,j})$ be the adjacency matrix of G, that is, A is a $d \times d$ symmetric matrix where $a_{i,j} = 1$ if $(i, j) \in E(G)$ and $a_{i,j} = 0$ if $(i, j) \notin E(G)$. Since G is Δ-regular and not bipartite, we know that the minimum eigenvalue of A satisfies $\lambda_{min} > -\Delta$. Hence, the matrix $A + \Delta I$ is positive semidefinite. Therefore

$$A + \Delta I = BB^T, \tag{1}$$

for some $d \times d$ matrix B. Let v_i be the ith row of the matrix B and set $V = \{v_1, \ldots, v_d\} \subset \mathbb{R}^d$. Reading off the entries of the matrix (1), we obtain

$$||v_i - v_j||^2 = (v_i - v_j) \cdot (v_i - v_j) = v_i \cdot v_i - 2v_i \cdot v_j + v_j \cdot v_j = 2\Delta - 2a_{i,j},$$

for $i \neq j$. In other words, $||v_i - v_j|| = \sqrt{2\Delta - 2a_{i,j}} \leq r = \sqrt{2\Delta - 1}$ if and only if $(i, j) \in E(G)$. $\square$

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