A note on the clique chromatic number of geometric graphs

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Abstract

Given a finite point set $V$ in $\mathbb{R}^d$ and a parameter $r > 0$, the corresponding geometric graph $G$ in $\mathbb{R}^d$ is the graph whose vertex set is $V \subset \mathbb{R}^d$, and vertices $u, v \in V$ are adjacent in $G$ if and only if $||u - v|| \leq r$. In this note, we study the clique chromatic number of geometric graphs, that is, the minimum number of colors required to color the vertices of a graph such that every maximal clique of size at least two is not monochromatic. Improving the results of McDiarmid, Mitsche, and Pralat, we show that if $G$ is a geometric graph in $\mathbb{R}^d$ then $\chi_c(G) \leq 2^{O(d)}$. From the other direction, we show that there are geometric graphs $G$ in $\mathbb{R}^d$ such that $\chi_c(G) > \Omega(d^{1/2} (\log d)^{-1/2})$.

∗Supported by a Packard Fellowship and by NSF CAREER award DMS-1352121.
†Supported by Swiss National Science Foundation grants 200020-162884 and 200021-165977.
‡Supported by NSF grant DMS-1800736, an NSF CAREER award, and an Alfred Sloan Fellowship.
1 Introduction

Given a simple graph $G = (V,E)$, a proper coloring of $G$ is a coloring of the vertices such that no two adjacent vertices have the same color. The chromatic number of $G$, denoted by $\chi(G)$, is the minimum number of colors required to properly color $G$. A clique coloring of $G$ is a coloring of the vertices such that no maximal clique of size at least two is monochromatic. The clique chromatic number of $G$, denoted by $\chi_c(G)$, is the minimum number of colors required to clique color $G$. Clearly we have $\chi_c(G) \leq \chi(G)$ for every graph $G$, and if $G$ is triangle-free, then we have $\chi(G) = \chi_c(G)$. The clique chromatic number has been studied in many papers, and we refer the interested reader to [2, 3, 6, 5].

McDiarmid, Mitsche, and Pralat [5] initiated the study of the clique chromatic number of geometric graphs in $\mathbb{R}^d$. Given a finite point set $V$ in $\mathbb{R}^d$ and a parameter $r > 0$, the corresponding geometric graph $G$ in $\mathbb{R}^d$ is the graph whose vertex set is $V \subset \mathbb{R}^d$, and vertices $u, v \in V$ are adjacent in $G$ if and only if $||u - v|| \leq r$, where $||u - v||$ denotes the Euclidean distance between $u$ and $v$. McDiarmid et al. [5] showed that if $G$ is a geometric graph in $\mathbb{R}^d$, then $\chi_c(G) \leq 2(\sqrt{d} + 1)^d$. In the other direction, they showed for each $d \geq 1$, there are geometric graphs $G$ in $\mathbb{R}^d$ such that $\chi_c(G) > \Omega(d^{1/4}(\log d)^{-1/2})$. In what follows, we improve both of these bounds.

Theorem 1.1. The clique chromatic number of any geometric graph $G$ in $\mathbb{R}^d$ satisfies

$$\chi_c(G) \leq 2^{O(d)}.$$

Theorem 1.2. For every integer $d \geq 1$, there exists a geometric graph $G_d$ in $\mathbb{R}^d$ whose clique chromatic number satisfies

$$\chi_c(G_d) \geq \Omega(d^{1/2}(\log d)^{-1/2}).$$

2 A new upper bound

Proof of Theorem 1.1. Let $G = (V,E)$ be a geometric graph in $\mathbb{R}^d$ with parameter $r > 0$, where $V \subset \mathbb{R}^d$. Let $B_0$ be a sufficiently large ball with radius $R$ such that $V \subset B_0$, and let $\mathcal{C} = \{B_1, \ldots, B_m\}$ be a maximum packing of $B_0$, where $B_i$ is a ball centered at $v_i \in \mathbb{R}^d$ and with radius $r/4$. Note that $\mathcal{C}$ is finite by volume considerations, and in fact, $m = |\mathcal{C}| \leq \left(\frac{R}{(r/4)}\right)^d$. For each $i$, let $B'_i$ and $B''_i$ be balls centered at $v_i$ with radius $r/2$ and $r$ respectively. Set $\mathcal{C}' = \{B'_1, \ldots, B'_m\}$ and $\mathcal{C}'' = \{B''_1, \ldots, B''_m\}$. Notice that for any point $x \in B_0$, there is a ball $B_i \in \mathcal{C}$ such that $||x - B_i|| < r/4$, which implies that $x \in B'_i$. In other words, $B_0 \subset \bigcup_{i=1}^m B'_i$.

Let $G''$ be the intersection graph of the balls in $\mathcal{C}''$, that is, $V(G'') = \mathcal{C}''$, and two elements $B''_i, B''_j \in \mathcal{C}''$ are adjacent in $G''$ if and only if $B''_i \cap B''_j \neq \emptyset$. Notice
that $G''$ has maximum degree at most $12^d$. Indeed, for fixed $i$, the number of balls from $\mathcal{C}''$ that intersect $B_i''$ is equal to the number of balls from $\mathcal{C}''$ that lie within $B(v_i, 3r)$, where $B(v_i, 3r)$ is the ball centered at $v_i$ with radius $3r$. This is at most the number of balls from $\mathcal{C}$ that can be packed in $B(v_i, 3r)$. Using that $\mathcal{C}$ is a packing of balls of radius $r/4$, the number of balls in $\mathcal{C}$ that lie entirely within $B(v_i, 3r)$ is at most

$$\text{Vol}(B(v_i, 3r))/\text{Vol}(B_i) = 12^d.$$ 

As the graph $G''$ has maximum degree at most $12^d$, it has chromatic number $k \leq 12^d + 1$. So there is a mapping $c : [m] \to [k]$ such that if $B_i''$ and $B_j''$ have a nonempty intersection, then $c(i) \neq c(j)$. Now we color the points in $V$ with $2k$ colors of the form $(j, t)$ with $1 \leq j \leq k$ and $t \in \{1, 2\}$, as follows. For $v \in V$, let $i(v)$ be the minimum index $i$ such that $v \in B_i'$. Let $S_i$ be the set of vertices $v \in V$ for which $i(v) = i$. Thus, $S_1, \ldots, S_m$ form a partition of $V$. If $|S_i| = 1$, then color the vertex in $S_i$ with color $(c(i), 1)$. If $|S_i| > 1$, then color one vertex (chosen arbitrarily) in $S_i$ with color $(c(i), 1)$, and color the rest of the vertices in $S_i$ with color $(c(i), 2)$. Since any two points in $V$ that are adjacent in $G$ have distance at most $r$, they either belong to the same part $S_i$, or belong to two different parts $S_i$ and $S_j$ such that $B_i'' \cap B_j'' \neq \emptyset$. Let $K$ be a maximal clique in $V$. If $K$ contains two vertices $u, v$ that belong to different parts $S_i$ and $S_j$, then by the coloring $c$, vertices $u$ and $v$ also get different colors by considering the first coordinate, so $K$ is not monochromatic. Otherwise, there is an $i$ such that $i(v) = i$ for all vertices $v$ in $K$. So, all vertices of $K$ are in the ball $B_i'$, which has diameter $r$. By maximality of $K$, all vertices of $V$ in $B_i'$ and, in particular, all vertices of $S_i$ are in $K$. Since there are two vertices in $S_i$ that receive different colors, $K$ is not monochromatic. We have thus shown that no maximal clique of $G$ with at least two vertices is monochromatic, so

$$\chi_c(G) \leq 2k \leq 2(12^d + 1).$$

This completes the proof. \qed

3 A new lower bound

For the proof of Theorem 1.2, we will need the following lemma.

**Lemma 3.1** (Alon, Ben-Shimon, Krivelevich [1]). There is a positive constant $C$ so that for every natural number $d$ there exists a regular triangle-free graph $G$ on $d$ vertices with independence number $\alpha(G) < C \sqrt{d \log d}$. Moreover, $G$ is $\Delta$-regular where $\Delta = \Theta(\sqrt{d \log d})$. 

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Proof of Theorem 1.2. Let \( d \geq 1 \) and let \( G \) be a \( \Delta \)-regular graph on the vertex set \( V(G) = \{1, \ldots, d\} \) meeting the requirements of Lemma 3.1. Since \( G \) is triangle-free, we have
\[
\chi_c(G) = \chi(G) \geq \frac{d}{\alpha(G)} \geq \Omega(\sqrt{d/\log d}).
\]

Set \( r = \sqrt{2\Delta - 1} \). We can assume that \( d \) is sufficiently large so that \( \sqrt{2\Delta - 2} > 0 \), since otherwise the statement easily follows. We now use an argument due to Frankl and Maehara [4] to construct a \( d \)-element point set \( V = \{v_1, \ldots, v_d\} \) in \( \mathbb{R}^d \) such that \( ||v_i - v_j|| \leq r \) if and only if \( (i, j) \in E(G) \).

Let \( A = (a_{i,j}) \) be the adjacency matrix of \( G \), that is, \( A \) is a \( d \times d \) symmetric matrix where \( a_{i,j} = 1 \) if \( (i, j) \in E(G) \) and \( a_{i,j} = 0 \) if \( (i, j) \not\in E(G) \). Since \( G \) is \( \Delta \)-regular and not bipartite, we know that the minimum eigenvalue of \( A \) satisfies \( \lambda_{\min} > -\Delta \). Hence, the matrix \( A + \Delta I \) is positive semidefinite. Therefore
\[
A + \Delta I = BB^T,
\]
for some \( d \times d \) matrix \( B \). Let \( v_i \) be the \( i \)th row of the matrix \( B \) and set \( V = \{v_1, \ldots, v_d\} \subset \mathbb{R}^d \). Reading off the entries of the matrix (1), we obtain
\[
||v_i - v_j||^2 = (v_i - v_j) \cdot (v_i - v_j) = v_i \cdot v_i - 2v_i \cdot v_j + v_j \cdot v_j = 2\Delta - 2a_{i,j},
\]
for \( i \neq j \). In other words, \( ||v_i - v_j|| = \sqrt{2\Delta - 2a_{i,j}} \leq r = \sqrt{2\Delta - 1} \) if and only if \( (i, j) \in E(G) \). \( \square \)

References


