Pseudorandom Ramsey Graphs

Jacques Verstraete, UCSD

jacques@ucsd.edu

Abstract

We outline background to a recent theorem [47] connecting optimal pseudorandom graphs to the classical off-diagonal Ramsey numbers $r(s,t)$ and graph Ramsey numbers $r(F,t)$. A selection of exercises is provided.

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1. Linear Algebra

If A is a square symmetric matrix, then the eigenvalues of A are real. When A is an n by n matrix, we denote them $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. If the corresponding eigenvectors forming an orthonormal basis are $e_1, e_2, \dots, e_n$, then for any $x \in \mathbb{R}^n$, we may write $x = \sum x_i e_i$ and

$$\langle Ax, x\rangle = \sum_{i=1}^n \lambda_i x_i^2 \text{ and } \langle Ax, y\rangle = \sum_{i=1}^n \lambda_i x_i y_i. \quad (1)$$

For any $i \in [n]$, note $x_i = \langle x, e_i\rangle$. Furthermore, A is diagonalizable: $A = X^{-1}\Lambda X$ where Λ has $\lambda_1, \lambda_2, \dots, \lambda_n$ on the diagonal and $e_1, e_2, \dots, e_n$ are the columns of X. Since trace is independent of basis, for $k \geq 1$:

$$\mathrm{tr}(A^k) = \mathrm{tr}(X^{-k}A^kX^k) = \mathrm{tr}(\Lambda^k) = \sum_{i=1}^n \lambda_i^k. \quad (2)$$

The combinatorial interpretation of $\mathrm{tr}(A^k)$ as the number of closed walks of length k in the graph G will be used frequently. When A is the adjacency matrix of a graph G, we write $\lambda_i(G)$ for the ith largest eigenvalue of A and

$$\lambda(G) = \max\{|\lambda_i(G)| : 2 \leq i \leq n\} \quad (3)$$

and we refer to the eigenvalues of the graph rather than the eigenvalues of A. For $X, Y \subseteq V(G)$, the number $e(X,Y)$ of ordered pairs (x,y) such that $x \in X$, $y \in Y$ and $\{x,y\} \in E(G)$ is given by the inner product $\langle Ax, y\rangle$

in (1) where x and y are the characteristic vectors of X and Y respectively – thus $x_v = 1$ if $v \in X$ and $x_v = 0$ otherwise. References on spectral graph theory include [15, 19, 33, 53].

1.1. (n, d, λ)-graphs

If A is doubly stochastic, then e_1 is the constant unit vector with eigenvalue equal to the common row sum. In particular, if A is the adjacency matrix of a d-regular graph G, then $\lambda_1(G) = d$ and if $\lambda(G) = \lambda$, then the graph is referred to as an (n, d, λ)-graph. The complete graphs K_n are $(n, n-1, 1)$-graphs and the complete bipartite graphs $K_{n,n}$ are $(2n, n, n)$-graphs.

Moore graphs. A Moore graph of girth five is a d-regular graph with d^2+1 vertices and no cycles of length three or four. The diameter of such graphs is two, and if A is an adjacency matrix for such a graph, then

$$A^2 = J - A + (d-1)I. \tag{4}$$

Here J is the all 1 matrix and I is the identity matrix, both with the same dimensions as A. Then $\lambda_i^2 + \lambda_i = d - 1$ for all $i \geq 2$ and so every eigenvalue other than the largest is $\frac{1}{2}(1 \pm \sqrt{4d-3})$. The Petersen graph is an example with $d = 3$ and the spectrum is $3^1 1^5 (-2)^4$. The non-existence of d-regular Moore graphs when $d \notin \{2, 3, 7, 57\}$ was proved by Hoffman and Singleton [36] by considering integrality of the spectrum of the graph. A brief survey of Moore graphs is found in [59].

Cayley graphs. Let Γ be a group and $S \subseteq \Gamma$ closed under inverse. A Cayley graph (Γ, S) has vertex set Γ where $\{x, y\}$ is an edge if $xs = y$ for some $s \in S$. Let $\chi_\gamma : \gamma \in \Gamma$ denote the characters of Γ, then the eigenvalues of the Cayley graph

$$\lambda_\gamma = \sum_{s \in S} \chi_\gamma(s). \tag{5}$$

For example, if χ is the quadratic character of $\mathbb{F}_q$, then the Paley graphs P_q have vertex set $\mathbb{F}_q$ where $q = 1$ mod 4 and where $\{x, y\}$ is an edge if $\chi(x - y) = 1$ – this is a Cayley graph with generating set S equal to the set of non-zero quadratic residues of $\mathbb{F}_q$, namely $S = \{x : \chi(x) = 1\}$. Then the trivial character gives $\lambda_1 = \frac{1}{2}(q-1)$ as an eigenvalue, and the rest are given by exponential sums. In particular

$$\lambda(P_q) = \max_{1 \leq j \leq q-1} \Big| \sum_{x \in S} e^{\frac{2\pi i j x}{q}} \Big|. \tag{6}$$

In particular, it was proved by Gauss [31] that $\lambda(P_q) = \frac{1}{2}(q^{\frac{1}{2}} - 1)$. More generally, Weil's character sum inequality [31] can be used to give an upper bound on $\lambda(G)$ when G is a Cayley graph.

Many more examples of (n, d, λ)-graphs are given in the survey of Krivelevich and Sudakov [41]. For our purposes, we occasionally consider induced subgraphs of (n, d, λ)-graphs which are almost regular and whose eigenvalues are controlled by interlacing.

1.2. Alon-Boppana Theorem

The infinite d-regular tree T_d is the universal cover of d-regular graphs, so for any d-regular graph G with n vertices, the number of closed walks of length $2k$ from x to x is at least the number of closed walks of length $2k$ from the root of T_d to the root of T_d. This in turn is at least[1]

$$\frac{1}{k}\binom{2k-2}{k-1} d(d-1)^{k-1}. \tag{7}$$

The total number of walks of length $2k$ in G is at most $d^{2k} + (n-1)\lambda^{2k}$ by the trace formula, so

$$d^{2k} + (n-1)\lambda^{2k} \geq \frac{n}{k}\binom{2k-2}{k-1} d(d-1)^{k-1}. \tag{8}$$

Using this inequality, one can obtain a lower bound on λ. For fixed d, by selecting k to depend appropriately on n in (8), we obtain the Alon-Boppana Theorem [49]:

Theorem 1.1. *Let $d \geq 1$. If G_n is a d-regular n-vertex graph then*

$$\liminf_{n \to \infty} \lambda(G_n) \geq 2\sqrt{d-1}. \tag{9}$$

A very short proof of the Alon-Boppana Theorem was found by Alon [49,50]. Ramanujan graphs are d-regular graphs with $\lambda \leq 2\sqrt{d-1}$, and were constructed by Lubotzky, Philips and Sarnak [43] and Margulis [46] as Cayley graphs, and more recently using polynomial interlacing by Marcus, Spielman and Srivastava [45]. It turns out that random d-regular graphs [60] are also Ramanujan graphs with high probability, as proved in a major work by Friedman [27].

1.3. Expander Mixing Lemma

Let G be an (n, d, λ)-graph. If $X, Y \subset V(G)$ are disjoint and x and y are their characteristic vectors, then

$$e(X, Y) = \langle Ax, y \rangle = \sum_{i=1}^{n} \lambda_i x_i y_i. \tag{10}$$

[1] This is the number of closed walks which return for the first time to the root at time $2k$.

If G is d-regular then $\lambda_1 = d$ and so by Cauchy-Schwarz:

$$|e(X,Y) - dx_1y_1| \leq \Big|\sum_{i=2}^{n} \lambda_i x_i y_i\Big| \quad (11)$$

$$\leq \lambda\Big(\sum_{i=2} x_i^2\Big)^{\frac{1}{2}}\Big(\sum_{i=2}^{n} y_i^2\Big)^{\frac{1}{2}}. \quad (12)$$

Since $x_1 = \langle x, e_1\rangle = n^{-\frac{1}{2}}|X|$ we get the expander mixing lemma of Alon [2]:

Theorem 1.2. *Let G be an (n, d, λ)-graph and let $X, Y \subseteq V(G)$. Then*

$$\Big|2e(X) - \tfrac{d}{n}|X|^2\Big| \leq \lambda|X|(1 - \tfrac{|X|}{n}) \quad (13)$$

and if $|X| = \alpha n$ and $|Y| = \beta n$ then

$$\Big|e(X,Y) - \tfrac{d}{n}|X||Y|\Big| \leq \lambda\sqrt{|X\|Y|(1-\alpha)(1-\beta)}. \quad (14)$$

For a real $\alpha \geq 0$, a graph G with $e(G)$ edges and density $p = e(G)/\binom{n}{2}$ is α-pseudorandom if for every $X \subseteq V(G)$,

$$\Big|e(X) - p\binom{|X|}{2}\Big| \leq \alpha|X|. \quad (15)$$

The maximum of the quantity on the left over all subsets X is sometimes called the discrepancy of G. Note that in the random graph $G_{n,p}$, the expected number of edges in X is

$$\mathbb{E}(e(X)) = p\binom{|X|}{2} \quad (16)$$

and the Chernoff Bound can be used to show $G_{n,p}$ is $O(\sqrt{pn})$-pseudorandom with high probability provided p is not too small or not too large. A survey of pseudorandom graphs is given by Krivelevich and Sudakov [41]. In particular, the expander mixing lemma shows that (n, d, λ)-graphs are λ-pseudorandom. Bollobás and Scott [11] showed that if an n-vertex α-pseudorandom graph has density p, where $\frac{2}{n} \leq p \leq 1 - \frac{2}{n}$, then $\alpha = \Omega(\sqrt{p(1-p)}n^{\frac{3}{2}})$. The following conjecture due to the author is open:

Conjecture A. *Let G be an n-vertex graph of density p, where $\frac{1}{n} \leq p < \frac{1}{2}$. Then for some $X \subseteq V(G)$,*

$$e(X) = p\binom{|X|}{2} + \Omega(p^{\frac{1}{2}}n^{\frac{3}{2}}). \quad (17)$$

We remark that with high probability, $G_{n,p}$ satisfies this conjecture. When $p = \frac{1}{2}$ and $n = 2m$ is even, the complete bipartite graph $K_{m,m}$ has density $p = \frac{1}{2} + o(1)$ yet every set X of vertices has $e(X) \leq \frac{1}{4}|X|^2 + O(n)$.

2. Extremal (n, d, λ)-graphs

2.1. Triangle-free (n, d, λ)-graphs

Mantel's Theorem [44] states that a triangle-free graph with $2n$ vertices has at most n^2 edges, with equality only for complete n by n bipartite graphs, which are triangle-free $(2n, n, n)$-graphs. In general, one may ask for triangle-free (n, d, λ)-graphs, where d, n and λ must satisfy (8), and in particular, $\lambda = \Omega(d^{\frac{1}{2}})$. This leads to an extremal problem: when $\lambda = cd^{\frac{1}{2}}$, what is the largest d for which there is a triangle-free (n, d, λ)-graph? If G is any triangle-free (n, d, λ)-graph with adjacency matrix A, then

$$0 = \mathrm{tr}(A^3) \geq d^3 - \lambda^3(n-1) \quad (18)$$

and so $d \leq \lambda(n-1)^{\frac{1}{3}}$. If $\lambda = O(d^{\frac{1}{2}})$, then this gives $d = O(n^{\frac{2}{3}})$. Alon [3] gave an ingenious construction of a triangle-free Cayley (n, d, λ)-graph with $d = \Omega(n^{\frac{2}{3}})$ and $\lambda = O(n^{\frac{1}{3}})$, showing the the above bounds are tight. Other constructions were given by Kopparty [39] and Conlon [17], which are triangle-free (n, d, λ)-graphs and shown to have $\lambda = O(n^{\frac{1}{3}}\log n)$.

Theorem 2.1. *There exist triangle-free (n, d, λ)-graphs with $\lambda = O(n^{\frac{1}{3}})$ and $d = \Omega(n^{\frac{2}{3}})$ as $n \to \infty$.*

Kopparty [39] gives a triangle-free (n, d, λ)-graph which is a Cayley graph with vertex set $\mathbb{F}_{2^t}^3$ and generating set

$$S = \{(xy, xy^2, xy^3) : \mathrm{Trace}(x) = 1\} \quad (19)$$

where $d = \Omega(n^{\frac{2}{3}})$ and $\lambda = O(d^{\frac{1}{2}})$, which is essentially as good as the graphs of Alon [3].

Similar arguments show that if k is odd and G is a C_k-free graph then $d = O(\lambda n^{\frac{1}{k}})$ and when $\lambda = O(d^{\frac{1}{2}})$ this gives $d = O(n^{\frac{2}{k}})$, and this too is tight [3, 41]. If F is a bipartite graph, then in many cases, the densest known n-vertex F-free graphs are graphs whose degrees are all close to some number d and with $\lambda = O(d^{\frac{1}{2}})$ – see Füredi and Simonovits [30] for a survey of bipartite extremal problems. A particular rich source of such graphs is the random polynomial model, due to Bukh and Conlon [13].

2.2. Clique-free (n, d, λ)-graphs

If G is a K_4-free graph, then the common neighborhood of any pair of adjacent vertices is an independent set. If G is an (n, d, λ)-graph, this implies from (18) and (30) that

$$\frac{d^3 - \lambda(n-1)}{e(G)} \leq \alpha(G) \leq \frac{\lambda n}{d + \lambda}. \quad (20)$$

Consequently, $d = O(\lambda^{\frac{1}{3}} n^{\frac{2}{3}})$. More generally, Sudakov, Szabo and Vu [55] showed

$$d = O(\lambda^{\frac{1}{s-1}} n^{1-\frac{1}{s-1}}). \tag{21}$$

when G is a K_s-free (n,d,λ)-graph. When $\lambda = O(d^{\frac{1}{2}})$ this gives

$$d = O(n^{1-\frac{1}{2s-3}}). \tag{22}$$

Constructing such optimal K_s-free graphs for $s \geq 4$ is considered to be one of the major open problems in the area.

Conjecture B. *For $s \geq 4$, there is a K_s-free (n,d,λ)-graph with $\lambda = O(d^{\frac{1}{2}})$ and $d = \Omega(n^{1-\frac{1}{2s-3}})$.*

Alon and Krivelevich [4] constructed K_{s+1}-free graphs as follows. Consider the one-dimensional subspaces of $\mathbb{F}_q^s$, and join two subspaces by an edge if they are orthogonal to get a d-regular n-vertex (pseudo)graph $G[q,s]$ with[2]

$$n = \begin{bmatrix} s \\ 1 \end{bmatrix} = \frac{q^s-1}{q-1} \tag{23}$$

$$d = \begin{bmatrix} s-1 \\ 1 \end{bmatrix} = \frac{q^{s-1}-1}{q-1}. \tag{24}$$

Let S be the set of self-orthogonal 1-dimensional subspaces. Then $G[q,s]$ does not contain K_{s+1} disjoint from S, since $s+1$ pairwise orthogonal one-dimensional subspaces outside S are not possible: if generators are $x_1, x_2, \dots, x_{s+1}$, then $c_1x_1 + \cdots + c_sx_{s+1} = 0$ with say $c_i \neq 0$, and then an inner product with x_i gives the result. Furthermore, it is not hard to see using

$$A^2 = \binom{s-1}{1} J + (d - \binom{s-1}{1}) I \tag{25}$$

that $\lambda^2 \leq (d - \left[\begin{smallmatrix} s-1 \\ 1 \end{smallmatrix}\right]) \leq d$. A computation shows $G[q,s] - S$ has average degree $d = \Omega(n^{1-\frac{1}{s-1}})$ and $\lambda(G[q,s] - S) = O(d^{\frac{1}{2}})$ by interlacing.

The above construction can be improved in a straightforward way to give a K_{s+1}-free (n,d,λ) graph with $d = \Omega(n^{1-\frac{1}{s}})$ and $\lambda = O(d^{\frac{1}{2}})$ – see Question 7. This matches a construction of Bishnoi, Ihringer and Pepe [8], who define a graph $\Gamma[q,s]$ using the bilinear form $Q(x,y) = \xi x_1 y_1 + \sum_{i=2}^{s+1} x_i y_i$ with ξ a quadratic non-residue in $\mathbb{F}_q$. Let χ be the quadratic character of $\mathbb{F}_q$. The vertex set of $\Gamma[q,s]$ is

$$\{\langle x \rangle \in \begin{bmatrix} \mathbb{F}_q^{s+1} \\ 1 \end{bmatrix} : \chi(Q(x,x)) = 1\} \tag{26}$$

and two one-dimensional subspaces $\langle x \rangle$ and $\langle y \rangle$ are adjacent if $\chi(Q(x,y)) = 1$. A key fact is that the neighborhood of a vertex in $\Gamma[q,s]$ induces $\Gamma[q,s-1]$, and $\Gamma[q,1]$ is empty with $\frac{q}{2}$ vertices. Then an induction shows $\Gamma[q,s]$ is K_{s+1}-free, and $\Gamma[q,s]$ is an (n,d,λ)-graph with $d = \Theta(q^{\frac{s-1}{2}}) = \Theta(n^{1-\frac{1}{s}})$ and $\lambda = \Theta(d^{\frac{1}{2}})$.

A geometric construction of K_{s+1}-free graphs $H[k,s]$ was given by Alon and Krivelevich [4]: the vertex set of $H[k,s]$ is $\{1,2,\dots,s\}^k$ and vertices x and y are adjacent if their Hamming distance[3] is larger than $k(1 - \frac{2}{s(s+1)})$. The n-vertex graphs $H[k,s]$ are K_{s+1}-free, but for every $\delta > 0$ there exists k_0 such that for $k \geq k_0$, every set X with

$$|X| > n^{1-\frac{2-\delta}{s^2(s+1)^2 \log s}} \tag{27}$$

induces a graph containing K_s. A careful analysis of the case $s = 2$ by Erdős [21] gives an explicit construction of a lower bound for $r(3,t)$, namely for each $\delta > 0$,

$$r(3,t) = \Omega(t^{\frac{\log 4}{\log \frac{27}{8}} - \delta}) \tag{28}$$

where the exponent is roughly 1.1369. A more careful analysis of the construction of Alon and Krivelevich [4] gives slightly better bounds than (27).

2.3. Quadrilateral-free graphs

The "orthogonality" graphs above can also be used to get extremal C_4-free graphs: the graph $G[q,3]$ is quadrilateral-free and $(q+1)$-regular with q^2+q+1 vertices. This graph has $q+1$ loops (corresponding to self-orthogonal subspaces), and removing the loops we obtain a quadrilateral-free graph G_q with q^2+q+1 vertices and $\frac{1}{2}q(q+1)^2$ edges. These are the orthogonal polarity graphs of Erdős, Rényi and Sós [23]. Füredi [28] proved that they are extremal C_4-free graphs when $q > 13$. Near extremal d-regular C_6-free and C_{10}-free graphs with $\lambda = O(d^{\frac{1}{2}})$ are constructed using generalized polygons – see [29, 42].

3. Pseudorandom Ramsey graphs

3.1. Independent sets

If G is an (n,d,λ)-graph with $d \geq 1$ and X is an independent set in G, then from expander mixing

$$|2e(X) - \tfrac{d}{n}|X|^2| \leq \lambda |X| (1 - \tfrac{|X|}{n}) \tag{29}$$

which gives the bound[4]

$$|X| \leq \frac{n\lambda}{d+\lambda}. \tag{30}$$

[2] These are the q-binomial coefficients. In general, $\left[\begin{smallmatrix} r \\ k \end{smallmatrix}\right]$ is the number of k-dimensional subspaces of an r-dimensional vector space over $\mathbb{F}_q$, which is $(q^r-1)(q^{r-1}-1)\dots(q^{r-k+1}-1)/(q^k-1)(q^{k-1}-1)\dots(q-1)$.

[3] The Hamming distance is $d(x,y) = |\{i : x_i \neq y_i\}|$.

[4] In fact if μ is the least eigenvalue, one can similarly get the Hoffman-Delsarte Bound $|X| \leq -n\mu/(d-\mu)$ – see Godsil and Newman [32] for this and other bounds.

Let $\alpha(G)$ be the largest size of an independent set in a graph G. We write $\alpha(X)$ for the largest independent set in a subset X of $V(G)$. It is known that in $G_{n,\frac{1}{2}}$ the largest eigenvalue is $\Theta(n)$ and $\lambda = \Theta(n^{\frac{1}{2}})$. The above bound $\alpha(G_{n,\frac{1}{2}}) = O(n^{\frac{1}{2}})$ whereas in fact $\alpha(G_{n,\frac{1}{2}}) = O(\log n)$. In fact $G_{n,p}$ up until now gives the best lower bounds on Ramsey numbers $r(s,t)$: using the K_s-free process [10] one gets for $s \geq 3$:

$$r(s,t) = \Omega\Big(\frac{t}{\log t}\Big)^{\frac{s+1}{2}} (\log t)^{\frac{1}{s-2}} \tag{31}$$

whereas the best current upper bound on $r(s,t)$ is

$$r(s,t) = O\Big(\frac{t^{s-1}}{(\log t)^{s-2}}\Big) \tag{32}$$

due to Shearer [51]. The eigenvalue bound (30) is tight – see Question 11. It is a major conjecture that $\alpha(P_q) = \mathrm{polylog}(q)$ when q is a prime. Assuming the GRH, one can show $\alpha(P_q) = \Omega((\log q)(\log\log q))$.

3.2. Counting independent sets

It turns out that counting independent sets is valuable for Ramsey Theory. By Turán's Theorem [57], every graph of maximum degree d has an independent set of size at least $t = \frac{n}{d+1}$, and the number of such independent sets is at least

$$\frac{1}{t!} n(n-d-1)(n-2d-2)\cdots = (d+1)^t. \tag{33}$$

In (n,d,λ)-graphs, it turns out there are few independent sets that are much larger than this. The following is a slight improvement [47] of a theorem of Alon and Rödl [5]:

Theorem 3.1. *Let G be an (n,d,λ)-graph with $d \geq 1$, and let $t \geq \frac{2n\log^2 n}{d}$ be an integer. Then the number of independent sets of size t in G is at most*

$$\Big(\frac{2e^2\lambda}{\log^2 n}\Big)^t. \tag{34}$$

Proof. Let $X \subset V(G)$, and let

$$Y = \{v \in V(G) : |N(v) \cap X| \leq \frac{d}{2n}|X|\}. \tag{35}$$

By the expander mixing lemma, a calculation shows $|X||Y| \leq \frac{4\lambda^2 n^2}{d^2}$ and therefore $|X \cap Y| \leq \frac{2\lambda n}{d}$.

Now suppose we have an independent set $I = \{v_1, v_2, \ldots, v_r\}$ and sets $X_1, X_2, \ldots, X_r$ with $X_1 = V$ and $r < t$. Define Y as above with $X = X_r$ and let $X_{r+1} = V \backslash N(I) \subseteq X_r$. Suppose we choose v to add to I, and note that $v \in X_{r+1}$. We aim to show that the number of choices of v is substantially less than the number of choices of v_r if r is a bit larger than $\frac{n}{d+1}$. There are at most $|X_{r+1} \cap Y| \leq \frac{2\lambda n}{d}$ choices of $v \in X_{r+1} \cap Y$. If $v \in X_{r+1} \backslash Y$, then $|X_{r+1}| \leq (1 - \frac{d}{2n})|X_r|$ by definition of Y, and so this cannot happen for more than

$$s = \Big\lceil \frac{2n\log n}{d} \Big\rceil = \Big\lceil \frac{t}{\log n} \Big\rceil \tag{36}$$

values of r. We conclude that the number of independent t-sets is at most

$$\frac{1}{t!}\binom{t}{s} n^s \Big(\frac{2\lambda n}{d}\Big)^{t-s}. \tag{37}$$

A calculation using Stirling's Formula gives the required bound. □

We remark that we can get away with a similar count of independent sets in n-vertex graphs G with average degree d and $\lambda(G) \leq \lambda$.

3.3. Main Theorem

The following theorem [47] gives a new approach to Ramsey Theory via pseudorandom graphs:

Theorem 3.2. *If there is an F-free (n,d,λ)-graph G where $\lambda \geq \frac{\log^2 n}{4e^2}$ and $t = \lceil \frac{2n\log^2 n}{d} \rceil$, then*

$$r(F,t) > \frac{n\log^2 n}{4e^2\lambda}. \tag{38}$$

Proof. We assume relevant quantities are integers for convenience. Select uniformly and randomly a subset X of G with each vertex having probability $p = \frac{\log^2 n}{4e^2\lambda}$. The expected number of independent sets of size t in X is

$$\Big(\frac{2e^2\lambda}{\log^2 n}\Big)^t p^t = 2^{-t}. \tag{39}$$

Since $\mathbb{E}(|X|) = pn$, there exists $X \subset V(G)$ such that $|X| \geq pn - 2^{-t}$ with $\alpha(X) < t$. A computation gives the claimed bound on $r(F,t)$. □

We note that the proof actually tells us that with high probability, a random subset of size about pn supplies the lower bound in (38). It would be very interesting to give explicit examples of this subset in an F-free (n,d,λ)-graph. We remark that we can get away with a similar result in n-vertex F-free graphs G with average degree d and $\lambda(G) \leq \lambda$.

4. Constructing Ramsey graphs

We now recall Conjecture B: suppose that we have a K_s-free (n,d,λ)-graph with $d = \Omega(n^{1-\frac{1}{2s-3}})$ and $\lambda = O(d^{\frac{1}{2}})$. Then for fixed $s \geq 3$ and with $t = 2n^{\frac{1}{2s-3}}\log^2 n$, Theorem 3.2 gives

$$r(s,t) = \Omega\Big(\frac{t^{s-1}}{\log^{2s-4} t}\Big) \tag{40}$$

as $t \to \infty$. We recall $r(s,t) \le \binom{s+t-1}{s-1}$ from the Erdős-Szekeres Theorem [25], which for fixed s and $t \to \infty$ gives

$$r(s,t) \le (1+o(1))\frac{t^{s-1}}{(s-1)!}. \tag{41}$$

Shearer [51] improved this bound to

$$r(s,t) = O\Big(\frac{t^{s-1}}{\log^{s-2} t}\Big). \tag{42}$$

Thus the preceding lower bound (40) is tight up to logarithmic factors in t. Using the constructions discussed in the section on clique-free pseudorandom graphs, one obtains via Theorem 3.2 for $s \ge 3$:

$$r(s,t) = \Omega\Big(\frac{t^{\frac{s}{2}}}{(\log t)^{s-2}}\Big) \tag{43}$$

which is slightly worse than the bound (31) given by the random K_s-free process.

For $F = C_k$ and k odd, optimal F-free (n,d,λ)-graphs exist [3], and match or improve on the lower bound for $r(C_k,t)$ given by the C_k-free process. When $F = C_k$ and k is even, it is a major open problem to determine the order of magnitude of the extremal numbers $\mathrm{ex}(n,F)$ — the best lower bounds are due to Lazebnik, Ustimenko and Woldar [42]; a survey is given in [59]. When $F = C_4$, the orthogonal polarity graphs G_q from Section 2.3 plus loops are (n,d,λ)-graphs with $n = q^2+q+1$, $d = q+1$ and $\lambda = q^{\frac{1}{2}}$. Mubayi and Williford [48] checked that $\alpha(G_q) = \Theta(q^{\frac{3}{2}})$ – in fact this matches the Hoffman-Delsarte Bound [36]. An upper bound $r(C_4,t) < (t+1)^2$ was given by Erdős, Faudree, Rousseau and Schelp [22], and improved to

$$r(C_4,t) = O\Big(\frac{t^2}{(\log t)^2}\Big) \tag{44}$$

by Szemerédi [56]. On the other hand, the orthogonal polarity graphs G_q provide $r(C_4,t) = \Omega(t^{\frac{4}{3}})$ whereas random graphs give $r(C_4,t) = \Omega(t^{\frac{3}{2}})$. The random C_4-free process analyzed by Bohman and Keevash [10] gives $r(C_4,t) = \Omega(t^{\frac{3}{2}}(\log t)^{\frac{1}{2}})$. If instead we use Theorem 3.2, then noting $\lambda(G_q) \sim q^{\frac{1}{2}}$ we get with $t \sim 8q(\log q)^2$:

$$r(C_4,t) = \Omega\Big(\frac{q^2}{q^{\frac{1}{2}}}(\log q)^2\Big) = \Omega(t^{\frac{3}{2}}(\log t)^{\frac{1}{2}}) \tag{45}$$

which matches the C_4-free process [10]. One of the notoriously difficult open conjectures of Erdős [16] is as follows:

Conjecture C. *For some $\epsilon, T > 0$, $r(C_4,t) \le t^{2-\epsilon}$ for all $t \ge T$.*

It is plausible that for some $\epsilon > 0$ and all $k \ge 3$, $r(C_k,t) \le r(C_{k-1},t)^{1-\epsilon}$ for large enough t, but this and the above conjecture seem far beyond currently available techniques.

4.1. Subdivisions and random blocks

Let F be a graph and let $\mathcal{P} = (P_1, P_2, \dots, P_k)$ be a partition of $E(F)$ into bipartite graphs with at least one edge each. Let $X = \{x_1, x_2, \dots, x_k\}$ be new vertices, and let $F_{\mathcal{P}}$ be the graph with $V(F_{\mathcal{P}}) = V(F) \cup X$ and edge set

$$E(F_{\mathcal{P}}) = \bigcup_{i=1}^{k} \{\{x_i, y\} : y \in V(P_i)\}. \tag{46}$$

Let $L(F)$ be the family of all graphs $F_{\mathcal{P}}$ taken over partitions $\mathcal{P}$ of $E(F)$ into paths with at least one edge each. For instance, when F is a triangle, then $L(F)$ consists of C_4 plus a pendant edge and C_6. If F is a pentagon then every member of $L(F)$ is a cycle of length at most ten plus a set of pendant edges. Let G be a bipartite graph with parts U and V containing no member of $L(F)$ and such that every vertex of V has degree d.

Theorem 4.1. *Let F be a graph and let G be an $L(F)$-free bipartite graph with parts U and V such that $|U| = m$ and $|V| = n$ and every vertex of V has degree d. If $dt > m + t\log_2 n$, then*

$$r(F,t) > n. \tag{47}$$

Proof. For each $u \in U$, let (A_u, B_u) be a random partition of $N_G(u)$, independently for $u \in U$. Let H be the random graph with $V(H) = V$ obtained by placing a complete bipartite graph with parts A_u and B_u inside $N_G(u)$ for each $u \in U$. It is evident that H is F-free since G is $L(F)$-free. Now we show every independent set in H has size less than t, which is sufficient to show $r(F,t) > n$. Let I be a set of t vertices in H. If $|I \cap N_G(u)| = t_u$ for $u \in U$, then

$$\mathbb{P}(e(I \cap N_G(u)) = 0) = 2^{1-t_u}. \tag{48}$$

Since the partitions (A_u, B_u) are independent over different $u \in U$ and the sum of t_u is at least dt,

$$\begin{aligned}\mathbb{P}(e(I) = 0) &= \prod_{u \in U} \mathbb{P}(e(I \cap N_G(u)) = 0) \\ &= \prod_{u \in U} 2^{1-t_u} \\ &\le 2^{m-dt}.\end{aligned}$$

There are $\binom{n}{t}$ choices of I, so the expected number of independent sets of size t in H is

$$\binom{n}{t} 2^{m-(q+1)t} \le 2^{t\log_2 n + m - dt}. \tag{49}$$

Since $dt > m + t\log_2 n$, this is less than 1, and so with positive probability, every independent set in H has less than t. □

Random block constructions were used to give lower bounds on certain hypergraph Ramsey numbers [38].

Odd cycles. Let G be a bipartite graph of girth at least twelve with parts U and V of sizes $m = (q+1)(q^8+q^4+1)$ and $n = (q^3+1)(q^8+q^4+1)$ such that every vertex of V has degree $q+1$ and every vertex of U has degree q^3+1 – these are the incidence graphs of generalized hexagons of order (q, q^3) (see [34, 58] or [7] page 115 Corollary 5.38 for details about these constructions). Then G is $L(F)$-free and applying Theorem 4.1:

$$r(C_5, t) = \Omega(t^{\frac{11}{8}}). \tag{50}$$

This is the first example of a graph F such that $r(F, t)$ has a higher exponent than that given by the random F-free process. In a similar way, when $F = C_7$, the Ree-Tits octagons [34, 58] supply requisite bipartite graphs G of girth at least sixteen with parts of sizes $m = (q+1)(q^9+q^6+q^3+1)$ and $n = (q^2+1)(q^9+q^6+q^3+1)$ and all vertices in the larger part of degree $q+1$. Via Theorem 4.1:

$$r(C_7, t) = \Omega(t^{\frac{11}{9}}). \tag{51}$$

Tetrahedron. If $F = K_4$, then a C_4-free bipartite graph G with parts of size n containing no 1-subdivision[5] of K_4 is $L(F)$-free. It is possible to show that $|E(G)| = O(n^{\frac{7}{5}})$. If there is a d-regular n by n bipartite graph G containing no 1-subdivision of K_4 with n vertices and with $d = \Omega(n^{\frac{2}{5}})$ even, then one can produce a random graph H where every vertex has degree $\Omega(n^{\frac{4}{5}})$ and which may be an (n, d, λ)-graph with $\lambda = O^*(n^{\frac{2}{5}})$ (see the case of triangles in Conlon [17]). Via Theorem 3.2, this would then show $r(4, t) = t^{3-o(1)}$. However, the best construction of an n-vertex graph with no subdivision of K_4 has only $O(n^{\frac{4}{3}})$ edges, coming from generalized quadrangles [58].

4.2. Explicit Off-Diagonal Ramsey graphs

Alon [3] gave an explicit construction of a triangle-free n-vertex graph with independence number $O(n^{\frac{2}{3}})$. This remains the best explicit construction for lower bounds on $r(3, t)$, giving $r(3, t) = \Omega(t^{\frac{3}{2}})$. A simple way to construct a triangle-free n-vertex graph with independence number $O(n^{\frac{2}{3}})$ was given by Pudlak, as follows.

Starting with the bipartite incidence graph of a projective plane of order q – this is a quadrilateral-free $(q+1)$-regular bipartite graph H with $n = q^2+q+1$ vertices in each part – we order the vertices in each part, say $u_1 < u_2 < \cdots < u_n$ in part U and $v_1 < v_2 < \cdots < v_n$ in part V, and then form a graph H^* as follows. We let $V(H^*) = E(H)$ and let $\{u, v\}, \{w, x\} \in E(H^*)$ if $u < w$ in U and $v < x$ in V and $\{v, w\} \in E(H)$. In other words, we join the first and last edge of a path of length three in H whose vertices in U and V are increasing. Kostochka, Pudlak and Rödl [40] refer to such a graph as a superline graph. The graph H^* has $(q+1)(q^2+q+1) = |E(H)| \sim n^{\frac{3}{2}}$ vertices.

The independence number $\alpha(H^*)$ we claim is at most $2n$. To see this, note that an independent set of $2n$ vertices in H^* corresponds to a set of $2n$ edges in H. Amongst those $2n$ edges is a cycle, and that cycle contains a path of length three in H whose vertices in U and V are increasing, contradicting the independence. Therefore $\alpha(H^*) \le 2n$. Furthermore, H^* contains no triangles, since a triangle in H^* corresponds to three edges $\{u_1, v_1\}, \{u_2, v_2\}$ and $\{u_3, v_3\}$ with $u_1 < u_2 < u_3$ and $v_1 < v_2 < v_3$ as well as the edges $\{v_1, u_2\}, \{v_1, u_3\}$ and $\{v_2, u_3\}$. However, then $\{v_1, u_2, v_2, u_3\}$ is the vertex set of a quadrilateral in H, a contradiction. So H^* is triangle-free. While H^* has roughly the same independence number as the triangle-free (n, d, λ)-graphs of Alon [3], it is not as good as those graphs in terms of spectral properties – see Question 20.

Similar explicit constructions are given by Kostochka, Pudlak and Rödl [40], which show:

$$r(4, t) = \Omega(t^{\frac{8}{5}}) \;\; r(5, t) = \Omega(t^{\frac{5}{3}}) \;\; r(6, t) = \Omega(t^2). \tag{52}$$

It should be noted that these are far from the lower bounds supplied by random graphs, which have much larger exponents. We also note that the graph H^* contains very large complete bipartite graphs, and seem not to be amenable to an effective application of Theorem 3.2.

5. Multicolor Ramsey

For a graph F and $t \ge 1$, let $r_k(F, t)$ denote the Ramsey number $r(F, F, \ldots, F, t)$ with k colors (so $k-1$ copies of F). We consider $r_3(K_3, t) = r(3, 3, t)$. Alon and Rödl [5] solved an old conjecture of Erdős and Sós [24] stating $\frac{r(3,3,t)}{r(3,t)} \to \infty$ as $t \to \infty$, as follows:

Theorem 5.1. $r(3, 3, t) = \Theta^*(t^3)$.

Proof. Take Alon's triangle-free (n, d, λ)-graph G_n which has $d = \Theta(n^{\frac{2}{3}})$ and $\lambda = \Theta(n^{\frac{1}{3}})$, and let G_n' be a copy of G_n with the vertices randomly permuted. We claim that with positive probability,

$$\alpha(G_n \cup G_n') = O^*(t^{\frac{1}{3}}). \tag{53}$$

[5] A 1-subdivision $S_1(F)$ of a graph F is the graph obtained by replacing each edge $\{u, v\}$ of F with a path uwv. such that the paths are required to be internally disjoint. Note $S_1(F)$ is bipartite and has $|V(F)| + |E(F)|$ vertices.

To see this, pick a set X of t vertices of $G_n \cup G'_n$, where t is as in Theorem 3.1, so that the number of independent sets of size t in G_n is at most

$$\Big(\frac{4e^2\lambda}{\log^2 n}\Big)^t.$$

The probability that a set like this is independent in both G_n and G'_n is

$$\Big(\frac{4e^2\lambda}{\log^2 n}\Big)^{2t} \cdot \binom{n}{t}^{-1}. \qquad (54)$$

Using standard estimates on binomial coefficients, if n is large enough, this is less than 1, and therefore with positive probability $G_n \cup G'_n$ has no independent set of size $t = O^*(n^{\frac{1}{3}})$. □

More precisely, the proof above shows

$$r(3,3,t) = \Omega\Big(\frac{t^3}{(\log t)^4}\Big). \qquad (55)$$

The proof is not too far from best possible, since it is possible to show that if G is a graph that is a union of two n-vertex triangle-free graphs with average degree $O(n^{\frac{2}{3}})$, then by adapting the proofs of Shearer [51]:

$$\alpha(G) = \Omega(n^{\frac{1}{3}} \log n) \qquad (56)$$

and consequently we could not get a better lower bound than $r(3,3,t) = \Omega((\frac{t}{\log t})^3)$ using the above approach. The general upper bound for $r(3,3,t)$ is based on results of Shearer [51], as follows:

$$r(3,3,t) = O\Big(\frac{t^3}{(\log t)^2}\Big). \qquad (57)$$

The following conjecture is plausible:

Conjecture D.

$$r(3,3,t) = \Theta\Big(\frac{t^3}{(\log t)^2}\Big). \qquad (58)$$

Using known constructions of (n,d,λ)-graphs, Alon and Rödl [5] gave lower bounds on $r_k(F,t)$ for a variety of bipartite graphs F. Quite remarkably, it turns out that for all $k \geq 4$,

$$r_k(C_4,t) = \Theta\Big(\frac{t^2}{(\log t)^2}\Big). \qquad (59)$$

A common generalization of Theorem 3.2 and the results of Alon and Rödl [5] was found by He and Wigderson [35].

5.1. Blowups

For $r \geq 1$, a graph G, and disjoint sets $X_v : v \in V(G)$, the r-blowup of G is a graph $G(r)$ with vertex set $V(G(r)) = \bigsqcup_{v\in V(G)} X_v$ and edges $\{x,y\} : x \in X_u, y \in Y_v, \{u,v\} \in E(G)$. Kim and Mubayi [5] showed that blowing up a triangle-free graph from the triangle-free process does slightly better in the implicit logarithmic factors since we can count the number of independent sets of a certain size in the blowup – see Question 15. In particular, one obtains (55) by taking a Ramsey graph G for triangles versus K_t – this has $\Theta(\frac{t^2}{\log t})$ vertices – and using $G(r)$ with $r = \lceil \frac{t}{\log t} \rceil$.

The method of blowing up is effective for F-free graphs when F is not bipartite, but does not work when F is bipartite (since indeed the blowup contains large complete bipartite graphs).

Exercises

1. Let k be a positive even integer. Prove that if A is a symmetric pointwise non-negative matrix, and x is a pointwise non-negative unit vector, then $\langle A^k x, x\rangle \geq \langle Ax, x\rangle^k$.[6]

2. Prove that $\lambda(P_q) = \frac{1}{2}(q^{\frac{1}{2}} - 1)$.

3. For d-regular Moore graphs of girth 5, show $d \in \{2,3,7,57\}$.

4. Let $n \geq 1$ and let $\frac{1}{n} \leq p < \frac{1}{2}$. Prove that with high probability, every $X \subseteq V(G_{n,p})$ has

$$\Big|e(X) - p\binom{|X|}{2}\Big| = \Theta(p^{\frac{1}{2}} n^{\frac{3}{2}}).$$

5. Suppose there is a triangle-free (n,d,λ)-graph and let k be a positive integer. Prove that there is a triangle-free $(kn, kd, k\lambda)$-graph.

6. Derive (9) from (8), and show that for $d \leq \frac{1}{2}n$, $\lambda = \Omega(d^{\frac{1}{2}})$ as $d \to \infty$.

7. Let $s \geq 2$. Prove that there is a K_{s+1}-free (n,d,λ)-graph with $d = \Omega(n^{1-\frac{1}{s}})$ and $\lambda = O(d^{\frac{1}{2}})$ using k-dimensional spaces in $\mathbb{F}_q^{ks+k-1}$.

8. Let q be a suitable[7] prime power and

$$S = \{(xy, xy^2, xy^3) : \tfrac{q}{3} < \mathrm{Trace}(x) < \tfrac{2q}{3}\} \subset \mathbb{F}_q^3 \backslash \{0\}.$$

Prove that the Cayley graph with vertex set $\mathbb{F}_q^3$ and generating set S is triangle-free and an (n,d,λ)-graph where $\lambda = O(d^{\frac{1}{2}} \log n)$.

9. Prove that $\lambda(G[q,3])) = q^{\frac{1}{2}}$ and $G[q,3]$ has $q+1$ loops. Then find $e(G_q)$ and $\lambda(G_q)$.

[6] It is more challenging to prove this "Matrix Hölder" inequality when k is odd.
[7] For which prime powers does the proof work?

10. Prove that (n,d,λ)-graphs are λ-pseudorandom.

11. Show that $\alpha(P_q) = \Theta(q^{\frac{1}{2}})$ when q is a square.

12. Verify for $d \ge \lambda \ge 1$ and $t \ge \frac{2n\log^2 n}{d}$ and $s = \lceil \frac{t}{\log n} \rceil$ that

$$\frac{1}{t!}\binom{t}{s} n^s \Big(\frac{2\lambda n}{d}\Big)^{t-s} \le \Big(\frac{2e^2\lambda}{\log^2 n}\Big)^t.$$

13. Compare lower bounds on $r(3,t)$ from (i) the triangle-free process [9, 26] with (ii) the lower bound from Theorem 3.2 applied to Alon's triangle-free (n,d,λ)-graph, and (iii) the local lemma [52] applied to $G_{n,p}$.

14. It is known [56] that $r(C_4,t) = O(\frac{t^2}{(\log t)^2})$. Prove the same upper bound for $r_k(C_4,t)$ for $k \ge 3$. Use the polarity graphs G_q to show for each $k \ge 4$ that $r_k(C_4,t) = \Theta(\frac{t^2}{(\log t)^2})$ as $t \to \infty$.

15. Let G be a triangle-free graph with n vertices and independence number t, and $r \ge 1$. Prove that the r-blowup $G(r)$ contains at most $\frac{1}{m!}\binom{n}{t}(tr)^m$ independent sets of size m. Use this to prove for all $k \ge 2$ that

$$r_k(3,t) = \Omega\Big(\frac{t^k}{(\log t)^{2k-2}}\Big).$$

16. Check for $k \ge 2$ that as $t \to \infty$

$$r_k(3,t) = O\Big(\frac{t^k(\log\log t)^{k-2}}{(\log t)^{k-1}}\Big).$$

17. Suppose there exists a triangle-free (n,d,λ)-graph with $d = \Theta(n^{\frac{2}{3}})$ and $\lambda = O(d^{\frac{1}{2}})$. Prove that for $k \ge 1$, there exists a triangle-free (n,d,λ)-graph with $d = \Theta(kn^{\frac{2}{3}})$ and $\lambda = O(kd^{\frac{1}{2}})$.

18. Prove that $r_k(C_4,t) = \Theta(\frac{t^2}{(\log t)^2})$ when $k \ge 4$.

19. Let H be an n by n bipartite graph with $\Theta(n^{\frac{5}{3}})$ edges and no $K_{3,3}$. Prove that the superline graph H^* is K_5-free and determine the order of magnitude of $\alpha(H^*)$.

20. Let H be the bipartite incidence graph of a projective plane. Determine the order of magnitude of $\lambda(H^*)$.

21. Prove that an $m \times n$ bipartite graph with no cycle of length at most six has at most $(mn)^{\frac{2}{3}} + m + n$ edges. Then show $\mathrm{ex}(n, S_1(K_4)) = O(n^{\frac{7}{5}})$.

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