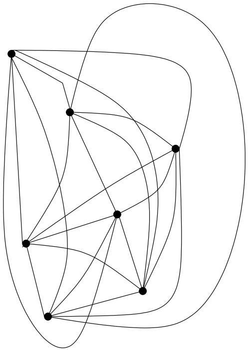
Short edges in complete topological graphs

Andrew Suk (UC San Diego)

November 9, 2023

Drawings of the complete graph K_n



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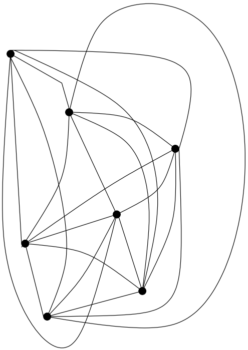

Question: Can we always find a “nice” planar subconfiguration?

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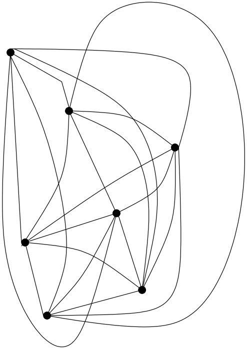

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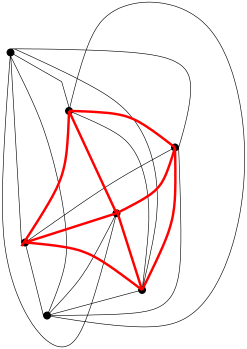

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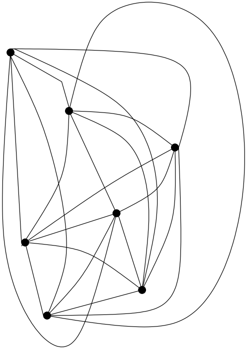

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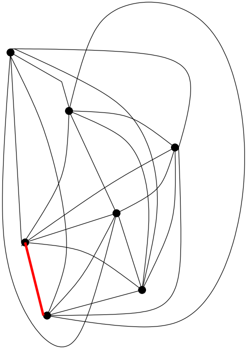

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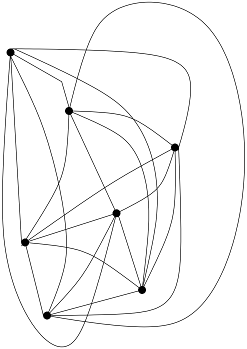

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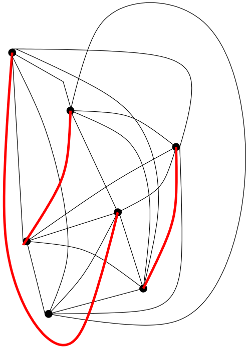

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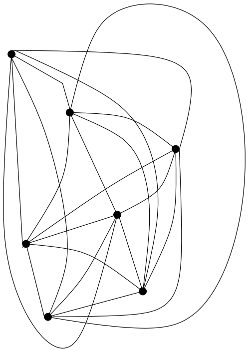

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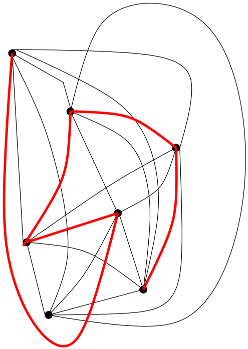

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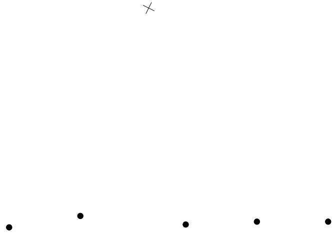

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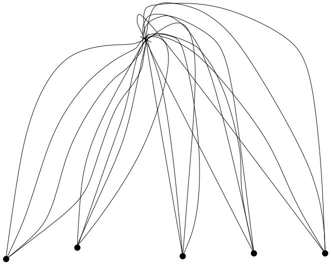

Every pair of edges cross.

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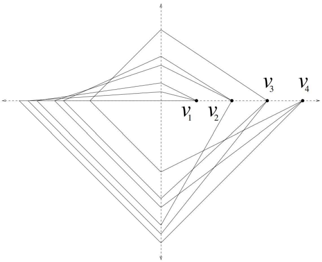
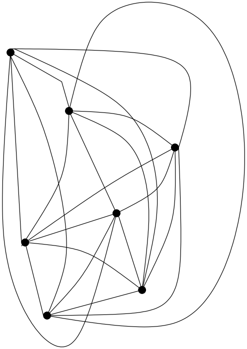


Pach-Tóth 2010. Every pair of edges cross, every pair of edges cross at most twice.

Simple condition is necessary



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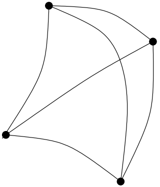
Every pair of edges cross at most once.

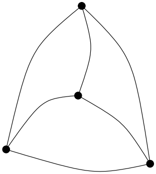
Simple Topological Graph $G = (V, E)$

V = points in the plane.

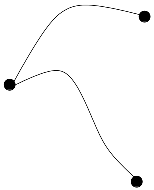
E = curves connecting the corresponding points (vertices).

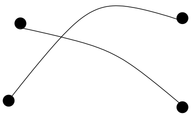
Every pair of edges have at most 1 point in common.

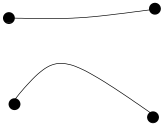




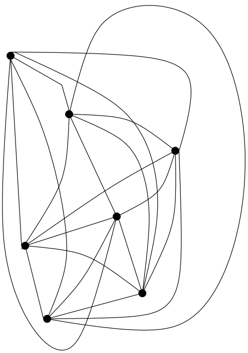
We will only consider simple topological graphs.







Complete simple topological graphs



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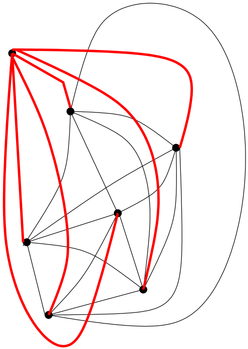

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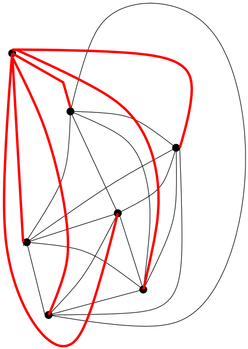

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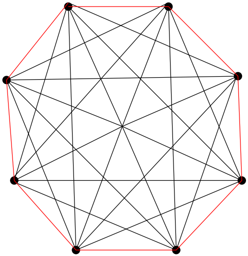

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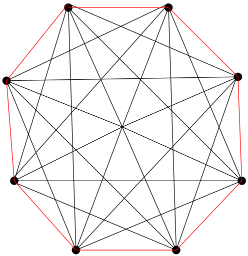

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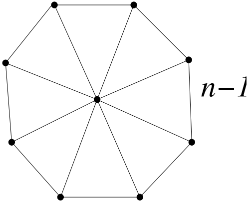

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Theorem (Harborth and Mengersen, 1994)

There are complete n-vertex simple topological graphs such that every edge crosses at least $(\frac{3}{4} + o(1))n$ other edges.

Finding an edge that crosses few other edges

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Conjecture (Brass, Moser, Pach, 2005)

$h(n) = o(n^2)$.

Informal definition. An edge is short if it crosses at most $o(n^2)$ other edges.

Short edges always exist in simple drawings of K_n

Theorem (Jan Kynčl, Pavel Valtr, 2009)

$$\Omega(n^{3/2}) < h(n) < O\left(\frac{n^2}{\log^{1/4} n}\right).$$

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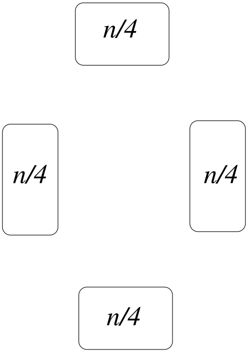

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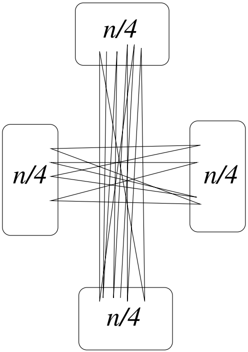

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Key ideas

1. VC-dimension theory
2. Minimality argument

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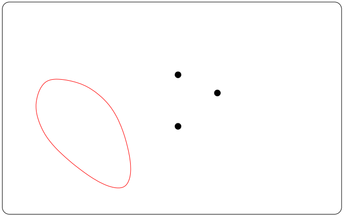
$\mathcal{F} =$ 

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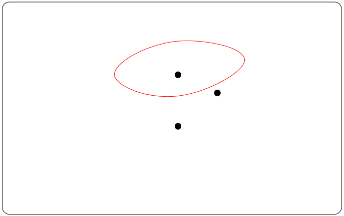

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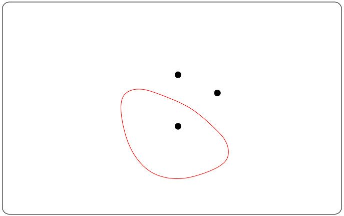

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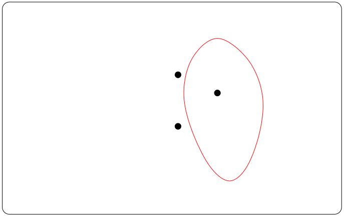

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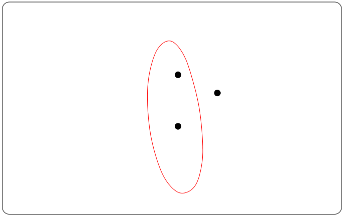
$\mathcal{F} =$ 

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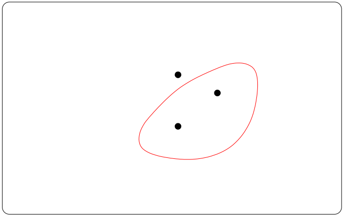
$\mathcal{F} =$ 

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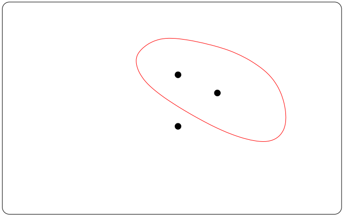
$\mathcal{F} =$ 

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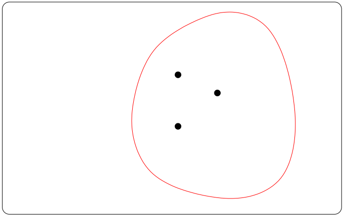

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The **VC-dimension of** $\mathcal{F}$ is the size of the largest subset $S \subset V$ that is shattered by $\mathcal{F}$.

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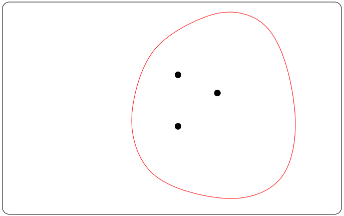

A more useful parameter

Dual VC-dimension. Let $\mathcal{F}$ be a set-system on a ground set V, $|V| = n$.

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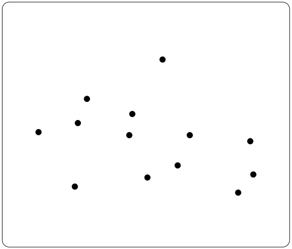

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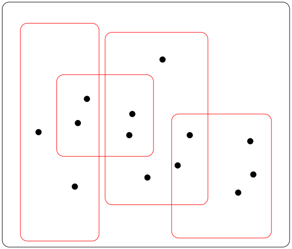
$\mathcal{F} =$ 

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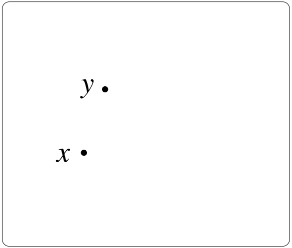

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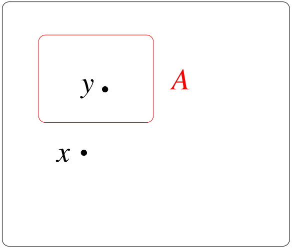
$\mathcal{F} =$ 

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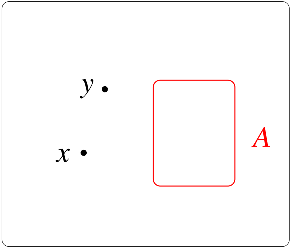
$\mathcal{F} =$ 

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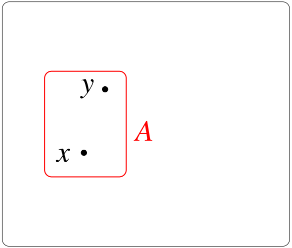
Definition: A set $A \in \mathcal{F}$ stabs $\{x, y\}$ if A contains exactly one of the two vertices.

$\mathcal{F} =$ 

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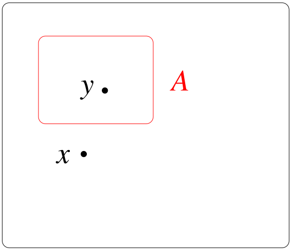

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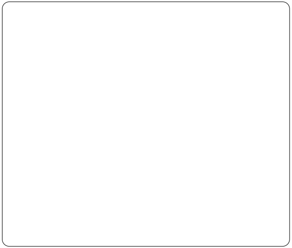
$\mathcal{F}$ is a set system on a ground set V with $\pi^*_{\mathcal{F}}(m) = O(m^d)$. Then there is a pair of vertices $x, y \in V$ such that $\{x, y\}$ is stabbed by at most $c|\mathcal{F}|/n^{1/d}$.

$\mathcal{F} =$ 

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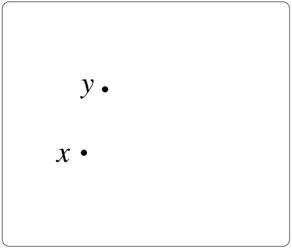

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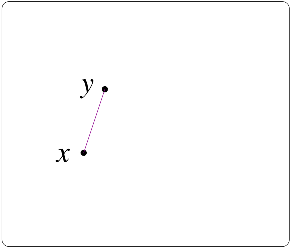
$\mathcal{F} =$ 

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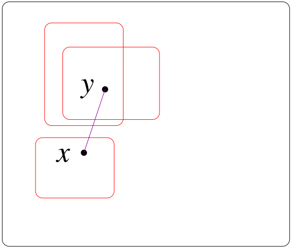
$\mathcal{F} =$ 

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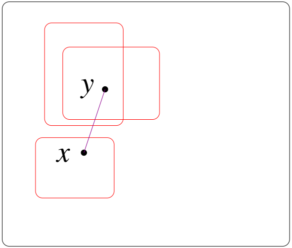
$\mathcal{F} =$ 

Together with an iterative re-weighting technique

Matching with low stabbing number

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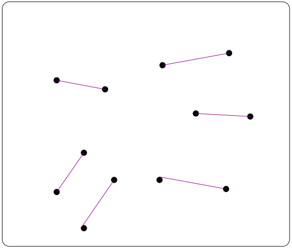
$\mathcal{F} =$ 

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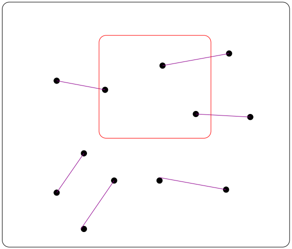
$\mathcal{F} =$ 

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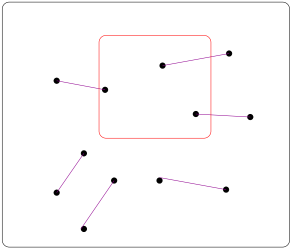
$\mathcal{F} =$ 

Combining Haussler's packing lemma + iterative re-weighting + triangle inequality

Lemma

$\mathcal{F}$ is a set system on V with $|V| = n$ and $\pi^*_{\mathcal{F}}(m) = O(m^d)$. Then there is a subset $X \subset V$, $|X| \leq O(n^{1/2+1/(2d)})$, and a perfect matching M on $V \setminus X$ such that

1. *Each $\{x, y\} \in M$ is stabbed by at most $O(|\mathcal{F}|/n^{1/(2d)})$ sets.*
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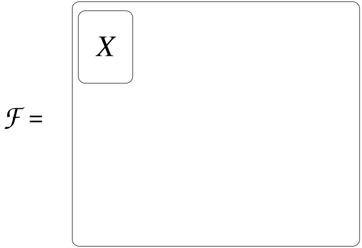

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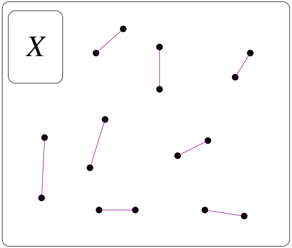
$\mathcal{F} =$ 

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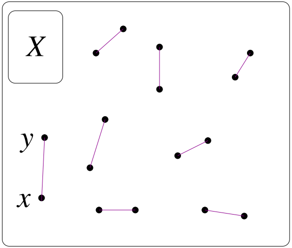

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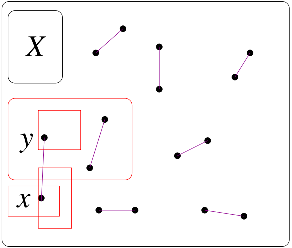
$\mathcal{F} =$ 

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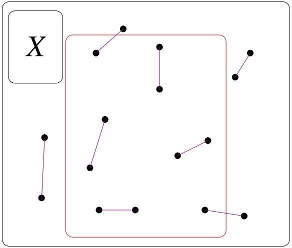
$\mathcal{F} =$ 

Proof of main result

Theorem (S., 2023+)

Every complete simple topological graph on n vertices contains an edge that crosses at most $O(n^{7/4})$ other edges.

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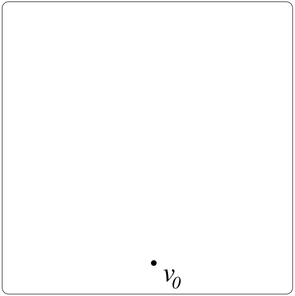

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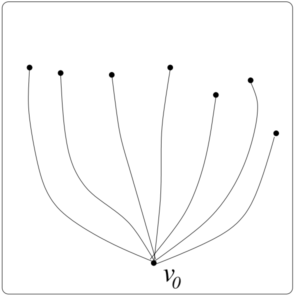

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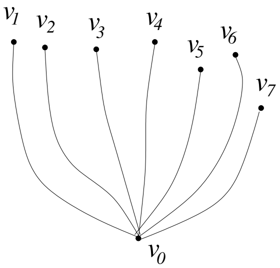

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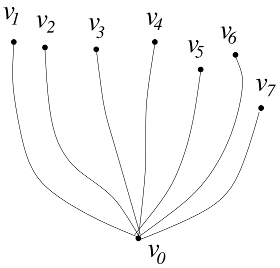

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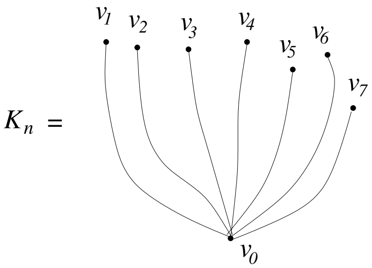

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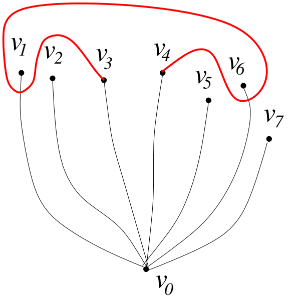

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Example: $T_{3,4} = \{v_1, v_6\}$.

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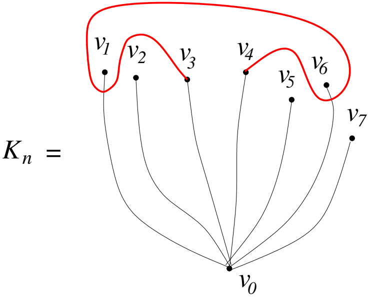

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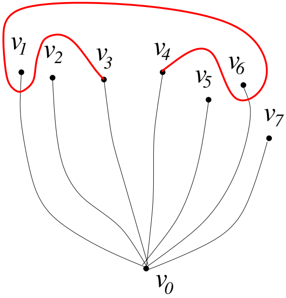

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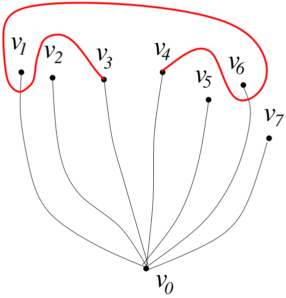

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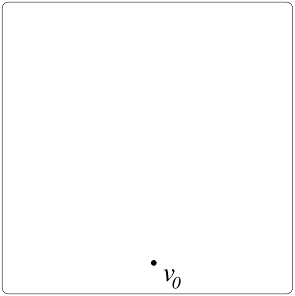
$K_n =$ 

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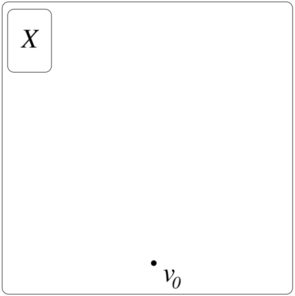

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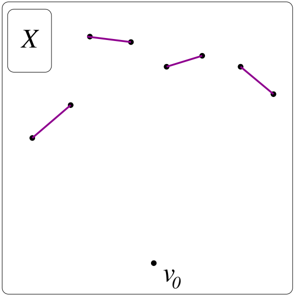

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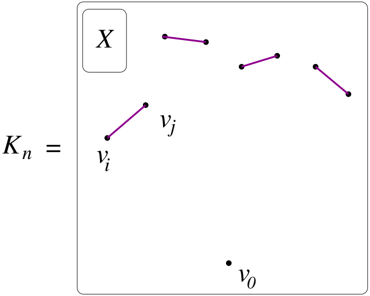

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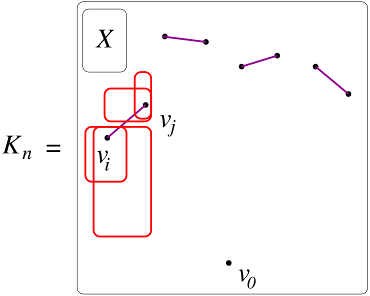

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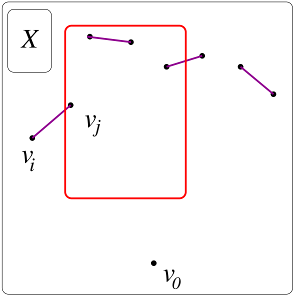
$K_n =$ 

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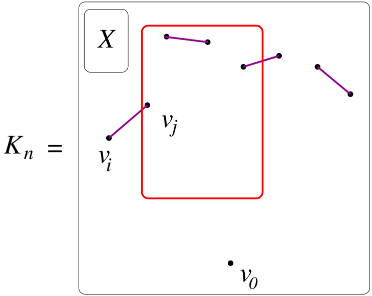

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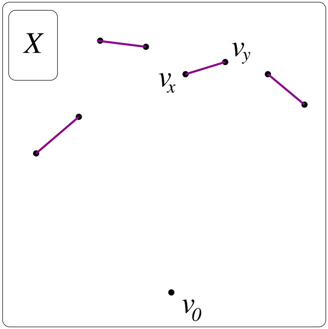

Let $\{v_x, v_y\}$ be the matching such that the triangle $T_{x,y} = (v_0, v_x, v_y)$ contains the fewest matchings.

Apply the matching with low stabbing number lemma

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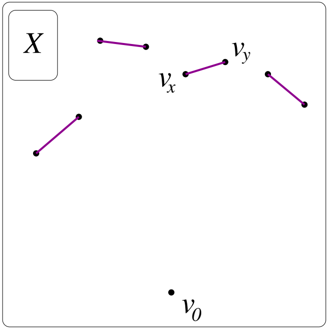

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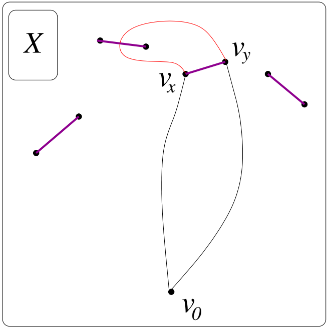
$K_n =$ 

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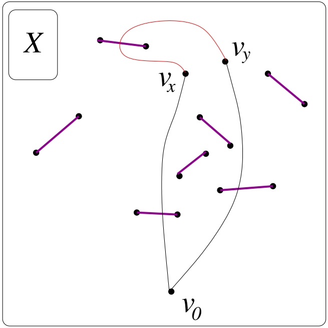

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Note. At most $O(n^{1/2})$ matchings stabs $T_{x,y}$.

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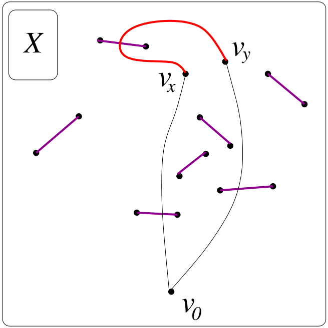

Claim. Edge $v_x v_y$ crosses at most $O(n^{7/4})$ other edges.

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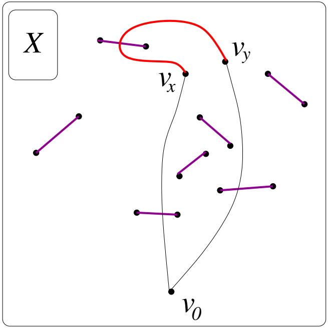
$K_n =$ 

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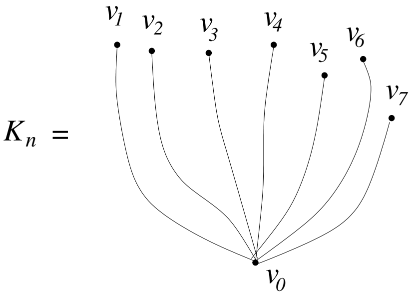
Cheat. $|x - y| < n^{3/4}$.

Apply the matching with low stabbing number lemma



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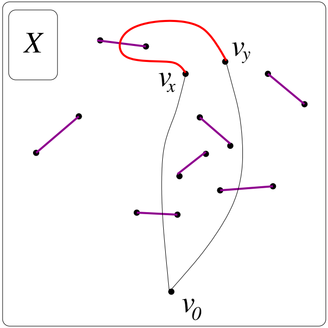

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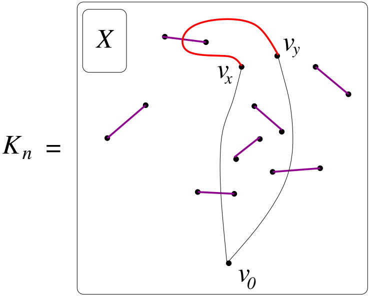
Counting edges crossing $v_x v_y$



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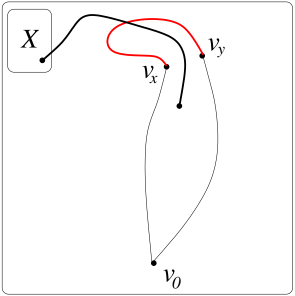

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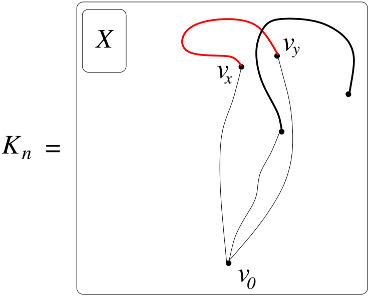
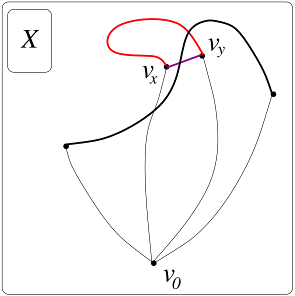


- E_0 edges incident to v_0. $|E_0| < n$.
- E_1 edges incident to X, $|X| < O(n^{3/4})$. $|E_1| = O(n^{7/4})$.
- E_2 edges with endpoint between v_x, v_y. $|E_2| = O(n^{7/4})$.

Counting edges crossing $v_x v_y$

K_n =



- E_0 edges incident to v_0. $|E_0| < n$.
- E_1 edges incident to X, $|X| < O(n^{3/4})$. $|E_1| = O(n^{7/4})$.
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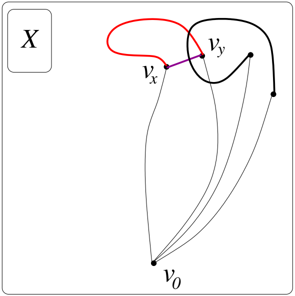

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Counting edges crossing $v_x v_y$

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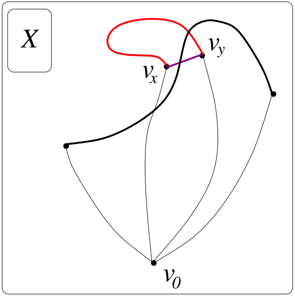
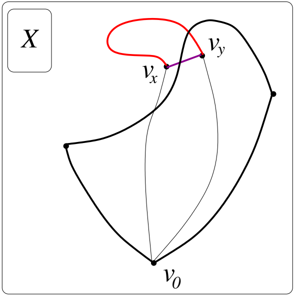


Observation. If v_i, v_j both lie outside (inside) of $T_{x,y}$ and $v_i v_j$ crosses $v_x v_y$, then $T_{i,j}$ stabs $\{v_x, v_y\}$.

Counting edges crossing $v_x v_y$

K_n =



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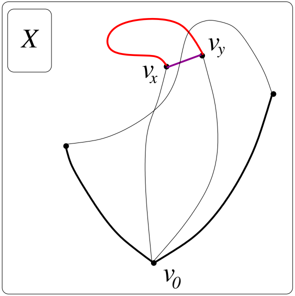
$K_n =$ 

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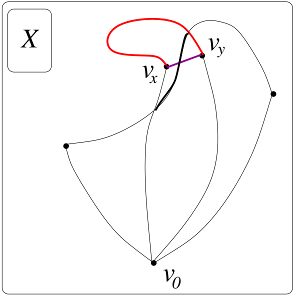

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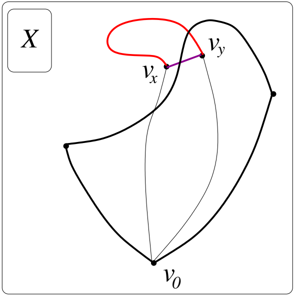

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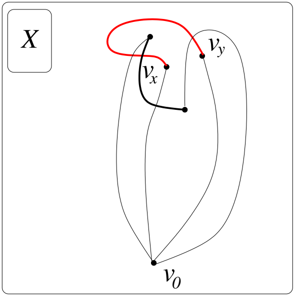

Observation. If v_i, v_j both lie outside (inside) of $T_{x,y}$ and $v_i v_j$ crosses $v_x v_y$, then $T_{i,j}$ stabs $\{v_x, v_y\}$.

Counting edges crossing $v_x v_y$

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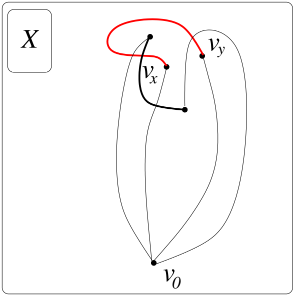
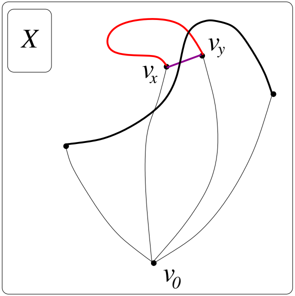


Observation. If v_i, v_j both lie outside (inside) of $T_{x,y}$ and $v_i v_j$ crosses $v_x v_y$, then $T_{i,j}$ stabs $\{v_x, v_y\}$.

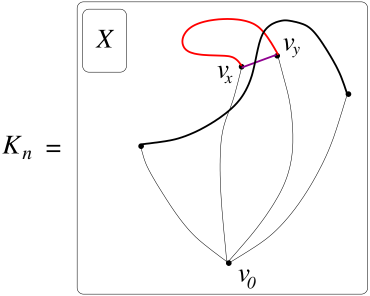
- E_3 remaining edges with both endpoints inside (outside) $T_{x,y}$ and crossing $v_x v_y$. $|E_3| = O(n^{7/4})$.

Counting edges crossing $v_x v_y$

$K_n =$ 

- E_0 edges incident to v_0. $|E_0| < n$.
- E_1 edges incident to X, $|X| < O(n^{3/4})$. $|E_1| = O(n^{7/4})$.
- E_2 edges with endpoint between v_x, v_y. $|E_2| = O(n^{7/4})$.
- E_3 both endpoints inside (outside) $T_{x,y}$. $|E_3| = O(n^{7/4})$.

Counting edges crossing $v_x v_y$



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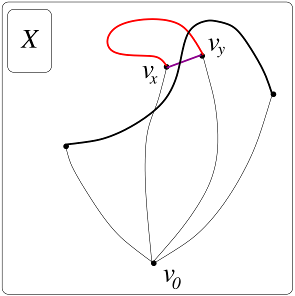
- E_3 both endpoints inside (outside) $T_{x,y}$. $|E_3| = O(n^{7/4})$.

Counting edges crossing $v_x v_y$

K_n =



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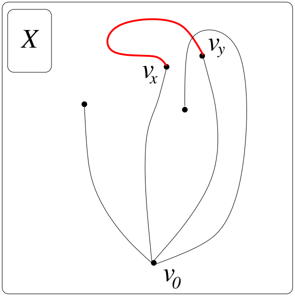
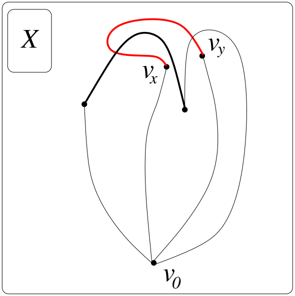
- E_2 edges with endpoint between v_x, v_y. $|E_2| = O(n^{7/4})$.
- E_3 both endpoints inside (outside) $T_{x,y}$. $|E_3| = O(n^{7/4})$.
- E_4 rest of the edges that crosses $v_x v_y$.

K_n =



Edges in E_4.

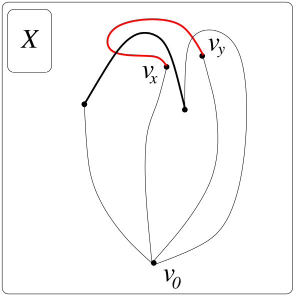
Counting edges crossing $v_x v_y$

$K_n =$ 

Edges in E_4.

Counting edges crossing $v_x v_y$

K_n =

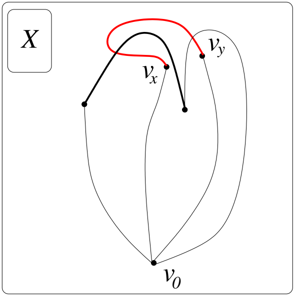


Edges in E_4.

Goal. $|E_4| = O(n^{7/4})$.

Counting edges crossing $v_x v_y$

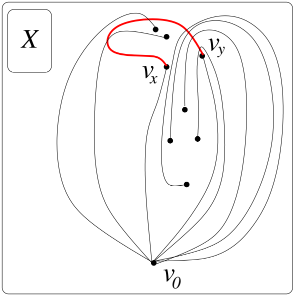
K_n =



Edges in E_4.

For sake of contradiction. If $|E_4| > cn^{7/4}$.

Counting edges crossing $v_x v_y$

$K_n =$ 

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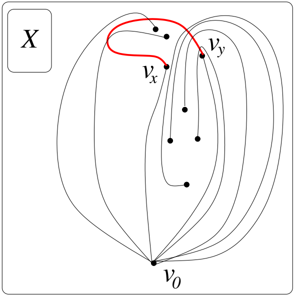
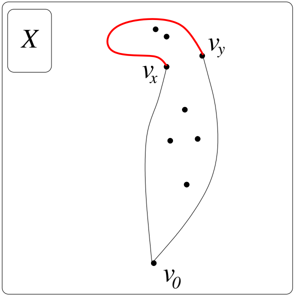
$K_n =$ 

Edges in E_4.

For sake of contradiction. If $|E_4| > cn^{7/4}$. At least $cn^{3/4}$ vertices "enter" triangle $T_{x,y}$.

Counting edges crossing $v_x v_y$

$K_n =$ 

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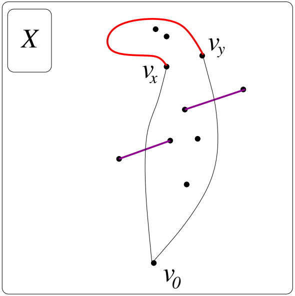

Edges in E_4.

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At most $O(n^{1/2})$ matchings stabs $T_{x,y}$.

Counting edges crossing $v_x v_y$

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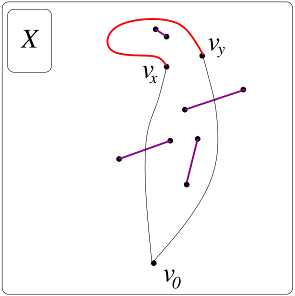

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At most $O(n^{1/2})$ matchings stabs $T_{x,y}$. Hence, many matchings lie inside.

Counting edges crossing $v_x v_y$

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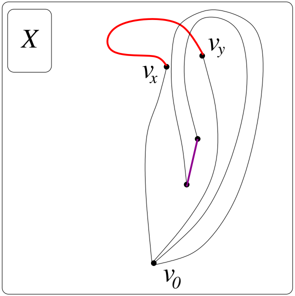

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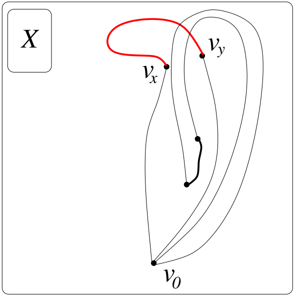

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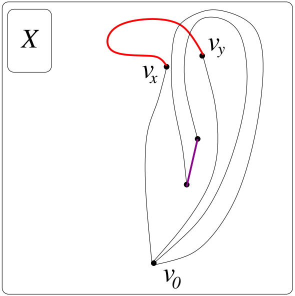
$K_n =$ 

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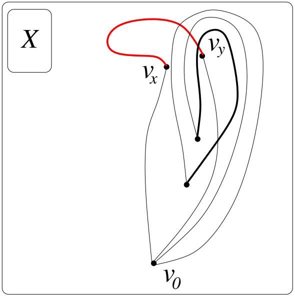
Counting edges crossing $v_x v_y$

K_n = 

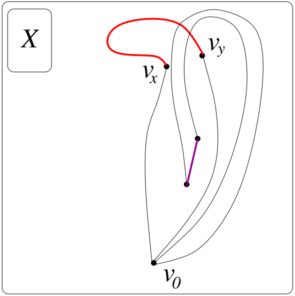
$(c/2)n^{3/4}$ **matchings** inside of $T_{x,y}$.

At most $O(n^{7/4})$ triangles $T_{i,j}$ stabs $\{v_x, v_y\}$.

Moreover: At most $O(n^{3/4})$ **matching** triangles $T_{i,j}$ stabs $\{v_x, v_y\}$.

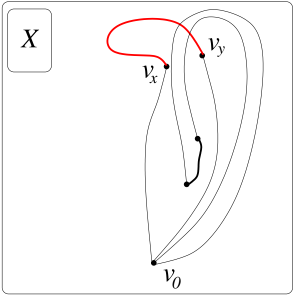
Counting edges crossing $v_x v_y$

$K_n =$



$(c/4)n^{3/4}$ matching whose corresponding topological edge must lie inside of triangle $T_{x,y}$.

Counting edges crossing $v_x v_y$

$K_n =$ 

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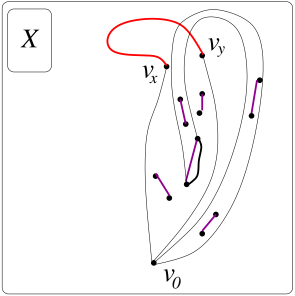

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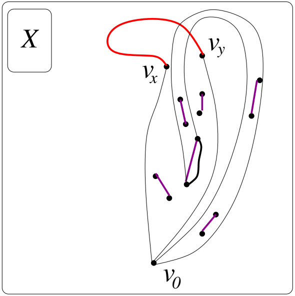
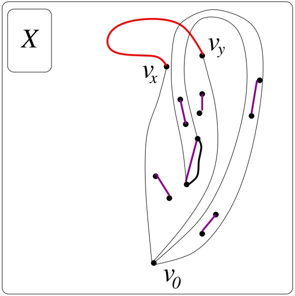


Punchline. One triangle $T_{i,j}$ will not contain $(c/10)n^{3/4}$ matchings from inside $T_{x,y}$.

Counting edges crossing $v_x v_y$

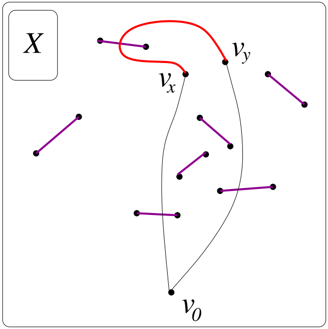
$K_n =$ 

Punchline. One triangle $T_{i,j}$ will not contain $(c/10)n^{3/4}$ matchings from inside $T_{x,y}$.

At most $O(n^{3/4})$ matchings lie inside $T_{i,j}$ and not in $T_{x,y}$.
Contradiction. $|E_4| = O(n^{7/4})$.

Putting it all together

K_n =



Edge $v_x v_y$ crosses at most

$$|E_0| + |E_1| + |E_2| + |E_3| + |E_4| = O(n^{7/4})$$

other edges.

$\square$.

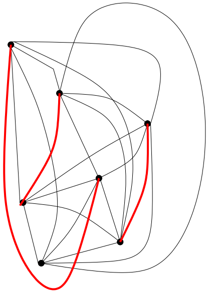
Open problems

$$\Omega(n^{3/2}) < h(n) < O(n^{7/4})$$

Conjecture (Kynčl-Valtr 2009, S. 2023+)

$h(n) = \Theta(n^{3/2})$.

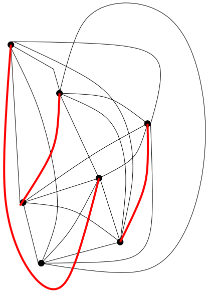
Open problem: Many pairwise disjoint edges



Theorem (Aichholzer-Garca-Tejel-Vogtenhuber-Weinberger 2022)

Every complete n-vertex simple topological graph contains $n^{1/2-o(1)}$ pairwise disjoint edges.

Open problem: Many pairwise disjoint edges

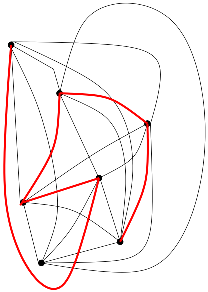


Theorem (Aichholzer-Garca-Tejel-Vogtenhuber-Weinberger 2022)

Every complete n-vertex simple topological graph contains $n^{1/2-o(1)}$ pairwise disjoint edges.

Rafla 1988. Noncrossing Hamiltonian cycle

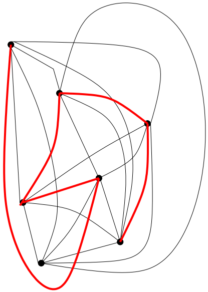
Open problem: Long noncrossing path



Theorem (S. 2023+)

Every complete n-vertex simple topological graph contains noncrossing path of length $\Omega(n^{1/9})$.

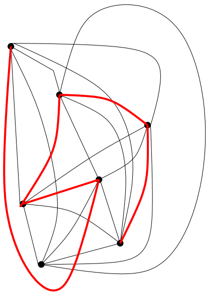
Open problem: Long noncrossing path



Theorem (S. 2023+)

Every complete n-vertex simple topological graph contains noncrossing path of length $\Omega(n^{1/9})$.

Previous best known bound. $\frac{\log n}{\log \log n}$ by Aichholzer et al., S.-Zeng.



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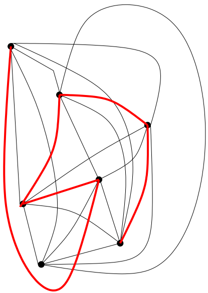

Theorem (S. 2023+)

Every complete n-vertex simple topological graph contains noncrossing path of length $\Omega(n^{1/9})$.

Problem. Noncrossing cycle of length $\Omega(n^{\epsilon})$.

Density type problems

Conjecture

Every n-vertex simple topological graph with εn^2 edges contains n^δ pairwise disjoint edges, where $\delta = \delta(\varepsilon)$.

Density type problems

Conjecture

Every n-vertex simple topological graph with εn^2 edges contains n^δ pairwise disjoint edges, where $\delta = \delta(\varepsilon)$.

Theorem (Fox-Pach-S., 2023+)

Every n-vertex simple topological graph with $\Omega(n^2)$ edges contains $n^{c/\log\log n}$ pairwise disjoint edges.

Previous best known bound. $(\log n)^{1+1/100}$ by Fox and Sudakov.

Thank you!