Semi-algebraic colorings of complete graphs

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June 19, 2019

Summary of results

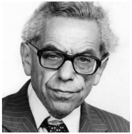
Joint work with Jacob Fox and János Pach

1. Tight bound for multicolor Ramsey numbers for semi-algebraic graphs.
2. Efficient regularity lemma for multicolor semi-algebraic graphs.
3. Tight bounds for generalized Ramsey numbers for semi-algebraic graphs.

Ramsey theory origins

Frank Ramsey, On a problem of formal logic (1930)

Paul Erdős and George Szekeres, A combinatorial problem in geometry (1935)



Ramsey theory origins

Issai Schur, Über die Kongruenz $x^m + y^m = z^m \mod p$ (1916)

Multicolor Ramsey numbers

Definition

For $m \geq 2$, The multicolor Ramsey number

$$r(\underbrace{3,\ldots,3}_{m \text{ times}})$$

is the minimum integer N such that for any m-coloring of the edges of K_N contains a monochromatic copy of K_3.

Known results

$$r(3,3) = 6 \qquad r(3,3,3) = 17 \qquad 51 \leq r(3,3,3,3) \leq 62$$

$$162 \leq r(3,3,3,3,3) \leq 307$$

$$2^m < r(\underbrace{3,\ldots,3}_{m \text{ times}}) < m!$$

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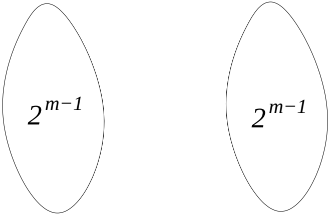

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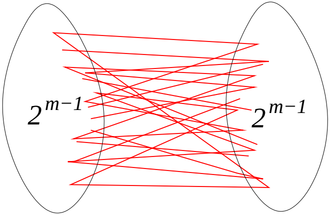

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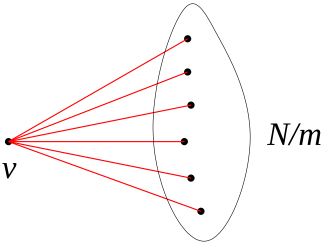


Best known bounds

Lower bound: Fredricksen-Sweet, Abbot-Moser.

Upper bound: Schur.

$$(3.199)^m < r(\underbrace{3,\ldots,3}_{m \text{ times}}) < 2^{O(m\log m)}$$

Erdős prize problems

Problem ($100)

Is the limit below finite or infinite?

$$\lim_{m\to\infty}\left(r(\underbrace{3,\ldots,3}_{m\text{ times}})\right)^{1/m}$$

Problem ($250)

Determine

$$\lim_{m\to\infty}\left(r(\underbrace{3,\ldots,3}_{m\text{ times}})\right)^{1/m}$$

Main result

If we insist that the m-coloring is **semi-algebraic** with bounded complexity:

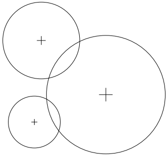
Theorem (Fox-Pach-S. 2019)

For $m \geq 2$,

$$r^{semi}(\underbrace{3, \ldots, 3}_{m \text{ times}}) = 2^{\Theta(m)}.$$

An example of a semi-algebraic graph

$V = \{A_1, ..., A_n\}$, n disks in the plane. $E = \{$pairs of disks that intersect$\}$.



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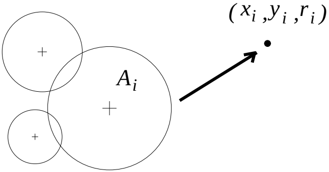

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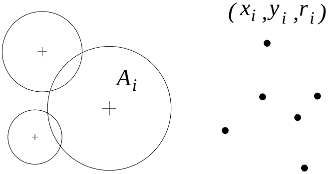

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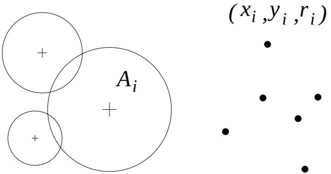

$A_i \to v_i = (x_i, y_i, r_i)$, $A_j \to v_j = (x_j, y_j, r_j)$. A_i and A_j cross if and only if

$$-x_i^2 + 2x_ix_j - x_j^2 - y_i^2 + 2y_iy_j - y_j^2 + r_i^2 + 2r_ir_j + r_j^2 \geq 0.$$

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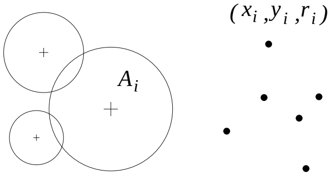

Graph $G = (V, E)$, $V = n$ points in $\mathbb{R}^3$

E defined by the polynomial

$$f(z_1, ..., z_6) = -z_1^2 + 2z_1z_4 - z_4^2 - z_2^2 + 2z_2z_5 - z_5^2 + z_3^2 + 2z_3z_6 + z_6^2.$$

$$(v_i, v_j) \in E \Leftrightarrow f(v_i, v_j) \geq 0.$$

We say that $G = (V, E)$ is a **semi-algebraic graph in d-space** if

$V = \{n \text{ points in } \mathbb{R}^d\}$

E defined by polynomials $f_1, ..., f_t$ and a Boolean formula Φ such that

$$(u, v) \in E$$

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$n \to \infty$

E **has bounded complexity:** d = dimension, t, and $deg(f_i)$ is bounded by some constant. (say ≤ 1000).

Main result

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For $m \geq 2$,

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$V = N$ points in $\mathbb{R}^d$

$\binom{V}{2} = E_1 \cup \cdots \cup E_m$,

each E_i is semi-algebraic with bounded complexity, contains a monochromatic K_3.

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For fixed $p \geq 3$ and $m \geq 2$,

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Sketch of the proof:

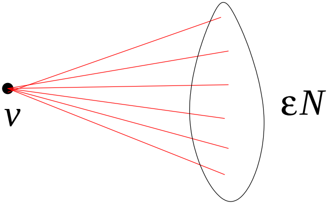
$$r^{semi}(\underbrace{3,\ldots,3}_{m \text{ times}}) \leq 2^{cm}, \qquad c = c(d,t)$$

Induction on m. Base case is trivial so we can assume m is large.

$$N = 2^{cm} \qquad V = N \text{ points in } \mathbb{R}^d \qquad \binom{V}{2} = E_1 \cup \cdots \cup E_m.$$

Sketch of the proof:

Goal: $\exists v \in V$ such that $|N_i(v)| \geq \epsilon N$ for some i.



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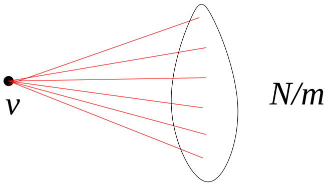
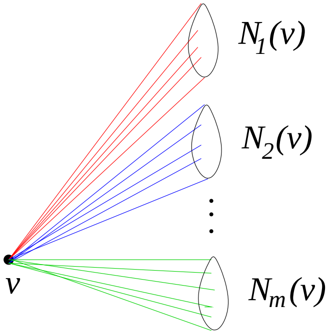
Not true: We can only assume $|N_i(v)| \geq N/m$ by pigeonhole.



Partition lemma (cutting lemma)

Consider the set system of neighborhoods: Ground set is V.

$$\mathcal{F} = \{N_i(v) : i \in [m], v \in V\}.$$



$\mathcal{F}$ controls all of the edges.

Partition lemma (cutting lemma)

Lemma

For $\mathcal{F}$ and parameter r, there is a partition on $V = V_1 \cup \cdots \cup V_r$, $|V_i| = N/r$, such that for each part V_i, ALL but at most $O(|\mathcal{F}|/r^{1/d})$ vertices $v \in V \setminus V_i$ will be monochromatic to V_i

$V =$ 

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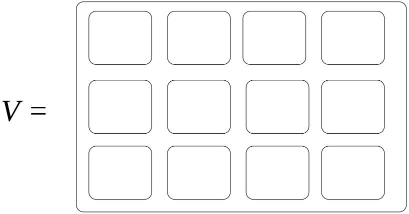

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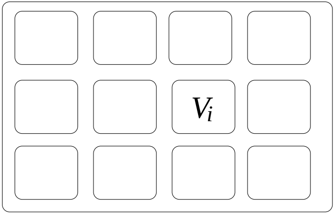
$V =$ 

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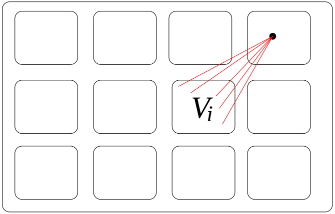
$V =$ 

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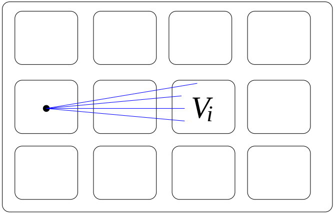
$V =$ 

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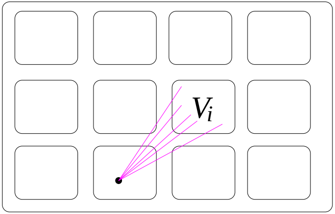
$V =$ 

Start: Set $\mathcal{F}_1 = \mathcal{F}$. $|\mathcal{F}_1| \leq Nm$. Apply the partition lemma to $\mathcal{F}_1$ with parameter $r = m^{2d}$.

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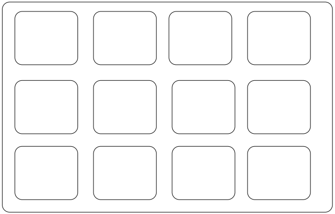

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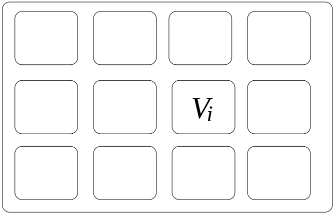
$V =$ 

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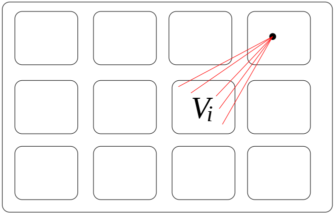
$V =$ 

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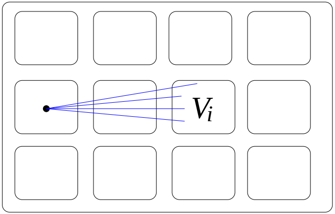
$V =$ 

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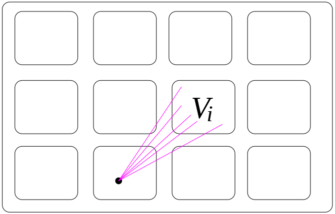
$V =$ 

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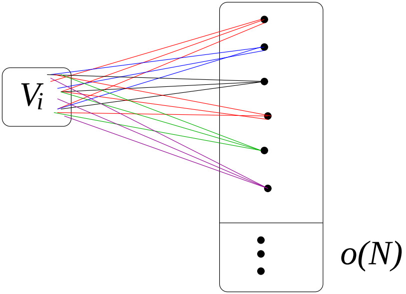

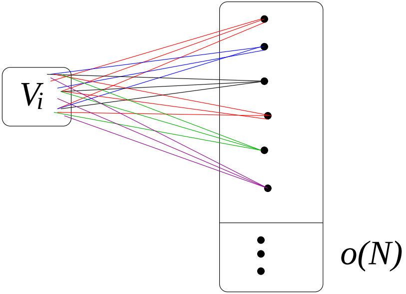

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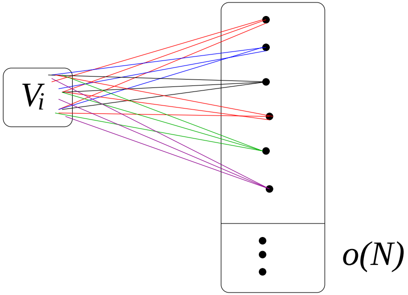
$$|V_i| \geq \frac{N}{m^{2d}} > 2^{c(m - \log m)}$$



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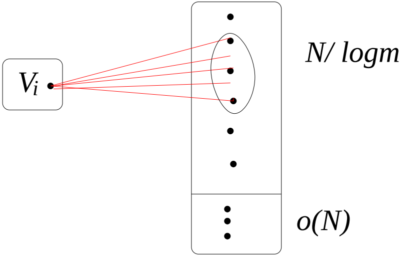

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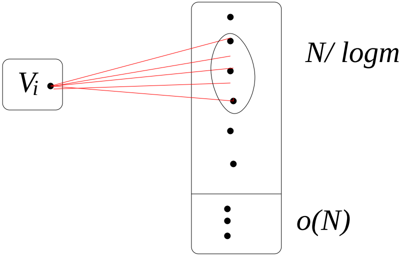

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$\mathcal{F}_2 \subset \mathcal{F}_1$, $|\mathcal{F}_2| \leq N \log m$.

$\mathcal{F}_2$ *controls* all but $o(N^2)$ edges.

Apply the partition lemma to $\mathcal{F}_2$ with parameter $r = \log^{2d} m$.

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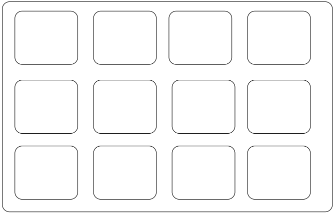
$V =$ 

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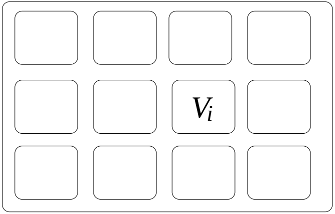
$V =$ 

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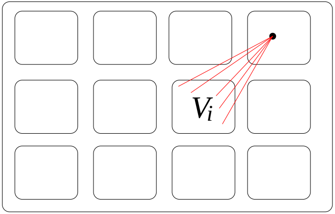
$V =$ 

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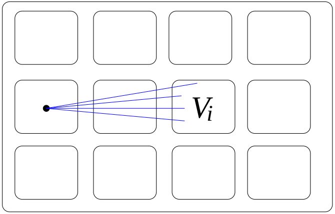
$V =$ 

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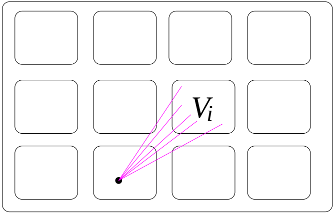
$V =$ 

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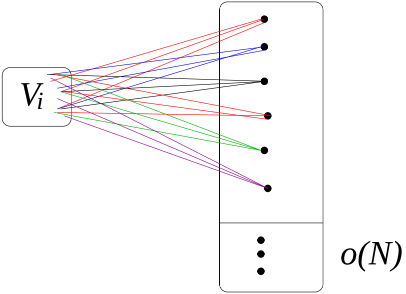

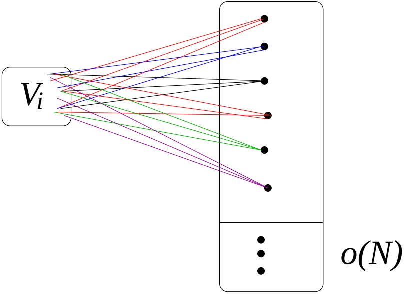

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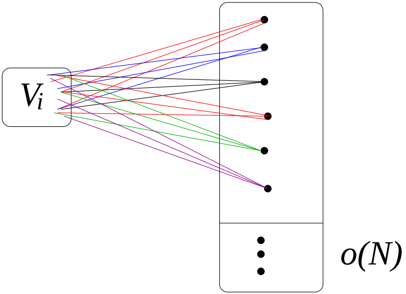

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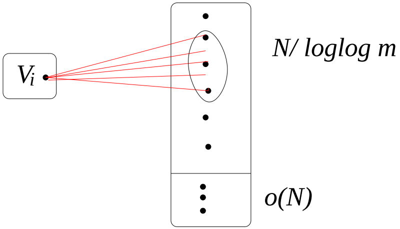

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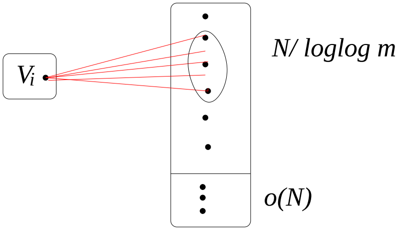
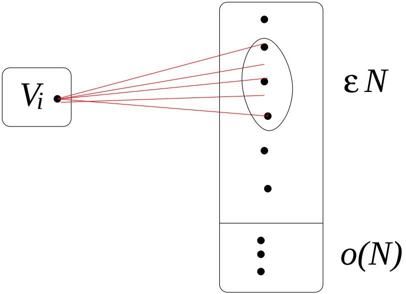


Case 2: There are fewer than $\log\log m$ distinct colors between V_i and the good vertices.

$\mathcal{F}_3 \subset \mathcal{F}_2, |\mathcal{F}_3| \leq N \log\log m$

$\mathcal{F}_3$ controls all but $o(N^2)$ edges.

After repeating this argument $\log^* m$ times:



Concluding remarks

1. Find some applications in geometry.
2. Goal: determine $r(\underbrace{3,\ldots,3}_{m \text{ times}})$.
3. Up coming paper (joint work with Jacob Fox and János Pach):

$$r(\underbrace{3,\ldots,3}_{m \text{ times}}) = 2^{\Theta(m)}$$

for graphs with bounded VC-dimension

Thank you!