Fun with Zeta of Graphs

Thank You!

Joint work with H. Stark, M. Horton, etc.
Labeling Edges of Graphs

We will use this labeling in the next section on edge zetas.

Let $X$ be a finite connected (not-necessarily regular) graph. Orient the $m$ edges. Label them as follows. Here the inverse edge has opposite orientation.

$e_1, e_2, \ldots, e_m,\ e_{m+1}=(e_1)^{-1}, \ldots, e_{2m}=(e_m)^{-1}$

Primes in Graphs

are equivalence classes $[C]$ of closed backtrackless tailless primitive paths $C$.

DEFINITIONS

- **backtrack**
  - equivalence class: change starting point
  - tail
  - Here $\alpha$ is the start of the path
  - non-primitive: go around path more than once
EXAMPLES of Primes in a Graph

E = CD
another prime \([C \times D]\), \(n=2, 3, 4, \ldots\)
ininitely many primes

\([C] = [e_1 e_2 e_3]\)
\([D] = [e_4 e_5 e_3]\)
\([E] = [e_1 e_2 e_3 e_4 e_5 e_3]\)
\(\nu(C) = 3, \nu(D) = 4, \nu(E) = 6\)

\(E = CD\)

Ihara Zeta Function – Unweighted
Possibly Irregular Graphs

\[\zeta_V(u, X) = \prod_{[C]} (1 - u^{\nu(C)})^{-1}\]

Ihara's Theorem (Bass, Hashimoto, etc.)
\(A = \) adjacency matrix of \(X\)
\(Q = \) diagonal matrix; \(j\)th diagonal entry
= degree \(j\)th vertex -1;
\(r = \) rank fundamental group = \(|E| - |V| + 1\)

\[\zeta_V(u, X)^{-1} = (1 - u^2)^r \det(I - Au + Qu^2)\]

Here \(V\) is for vertex
What happens for weighted graphs?

If each oriented edge \( e \) has weight \( \nu(e) \), define length of path \( C = e_1 \cdots e_s \) as
\[
\nu(C) = \nu(e_1) + \cdots + \nu(e_s).
\]
Just plug this \( \nu \) into the definition of zeta.
Call it \( \zeta(u, X, \nu) \)

Question: For which weights do we get an Ihara formula?

Remarks for \( q+1 \)-Regular Unweighted Graphs Mostly

- Riemann Hypothesis, (non-trivial poles on circle of radius \( q^{-1/2} \) center 0), means graph is Ramanujan i.e., non-trivial spectrum of adjacency matrix is contained in the interval \((-2\sqrt{q}, 2\sqrt{q}) = \) spectrum for the universal covering tree [see Lubotzky, Phillips & Sarnak, Combinatorica, 8 (1988)].

- Ihara zeta has functional equations relating value at \( u \) and \( 1/(qu) \), \( q=\)degree - 1
Set \( u=q^{-s} \) to get \( s \) goes to \( 1-s \).
### The Prime Number Theorem (irregular unweighted graphs)

- \( \pi_X(m) \) = number of primes \([C]\) in \(X\) of length \(m\)
- \( \Delta \) = g.c.d. of lengths of primes in \(X\)
- \( R = \) radius of largest circle of convergence of \(\zeta(u,X)\)

If \( \Delta \) divides \(m\), then

\[
\pi_X(m) \sim \Delta^R \frac{R^{-m}}{m}, \text{ as } m \to \infty.
\]

The proof comes from exact formula for \(\pi_X(m)\) by analogous method to that of Rosen, *Number Theory in Function Fields*, page 56.

- \( N_m = \# \text{ closed paths of length } m \text{ with no backtrack, no tails} \)

\[
\frac{d}{du} \log \zeta(u,X) = \sum_{m=1}^{\infty} N_m u^m
\]

### What about PNT for graph \(X\) with positive integer weights \(\nu\)?

You can inflate edge \(e\) by adding \(\nu(e)-1\) vertices. New graph \(X_\nu\) has determinant formulas and PNT similar to previous.

Some things do change:
- e.g. size of adjacency matrix, exact formula.
2 Examples
$K_4$ and $X=K_4$-edge

$\zeta_V^{-1}(u, K_4) = \frac{1}{(1-u^2)^2(1-u)(1-2u)(1+u+2u^2)^3}$

$\zeta_V^{-1}(u, X) = \frac{1}{(1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3)}$

For weighted graphs with non-integer wts, $1/\zeta$ not a polynomial

$m$ for the examples

$x \frac{d}{dx} \log \zeta(x, K_4) = 24x^3 + 24x^4 + 96x^6 + 168x^7 + 168x^8 + 528x^9 + O(x^{10})$

$\pi(3)=8 \quad \pi(4)=6 \quad \pi(5)=0$

$N_6 = \sum_{d|6} d \pi(d) = \pi(1) + 2 \pi(2) + 3 \pi(3) + 6 \pi(6)$

$\pi(6) = 24$

$x \frac{d}{dx} \log \zeta(x, K_4-e) = 12x^3 + 8x^4 + 24x^6 + 28x^7 + 8x^8 + 48x^9 + O(x^{10})$

$\pi(3)=4 \quad \pi(4)=2 \quad \pi(5)=0 \quad \pi(6)=2$
Poles of Zeta for $K_4$ are
$$\{1,1,1,-1,-1,\frac{1}{2},r_+,r_+,r_-,r_-,r_-,r_+\}$$

where $r_{\pm}=(-1\pm\sqrt{-7})/4$ and $|r|=1/\sqrt{2}$
$\frac{1}{2}$=Pole closest to 0 - governs prime number thm

Poles of zeta for $K_4-e$ are
$$\{1,1,-1,i,-i,r_+,r_-,\alpha,\beta,\beta\}$$

$R = \alpha$ real root of cubic $\cong 0.6573$
$\beta$ complex root of cubic

Derek Newland's Experiments

Mathematica experiment with random 53-regular graph - 2000 vertices

Spectrum adjacency matrix $\zeta(52^{-s})$ as a function of $s$

Top row = distributions for eigenvalues of A on left and
Imaginary parts of the zeta poles on right $s=\frac{1}{2}+it$.
Bottom row contains their respective normalized level spacings.
Red line on bottom: Wigner surmise, $y = (\pi x/2)e^{-\pi x^2/4}$. 

2/7/2008
What are Edge Zetas?

Orient the edges of the graph. Recall the labeling!

Define Edge matrix $W$ to have $a,b$ entry $w_{ab}$ in $C$ & set $w(a,b)=w_{ab}$ if the edges $a$ and $b$ look like those below and $a \neq b^{-1}$.

Otherwise set $w_{ab}=0$. $W$ is $2|E| \times 2|E|$ matrix.

If $C = a_1a_2 \ldots a_s$ where $a_j$ is an edge, define edge norm to be $N_E(C) = w(a_s,a_1)w(a_1,a_2)w(a_2,a_3) \ldots w(a_{s-1},a_s)$.

Edge Zeta

$\zeta_E(W,X) = \prod_{(C) \text{ prime}} (1 - N_E(C))^{-1}$
Properties of Edge Zeta

- Set all non-0 variables, \( w_{ab} = u \) in the edge zeta & get Ihara zeta.
- Cut an edge, compute the new edge zeta by setting all variables equal to 0 if the cut edge or its inverse appear in subscripts.
- Edge zeta is the reciprocal of a polynomial given by a much simpler determinant formula than the Ihara zeta.
- Better yet, the proof is simpler (compare Bowen & Lanford proof for dynamical zetas) and Bass deduces Ihara from this.

\[ \zeta_E(W, X) = \det(I - W)^{-1} \]

Determinant Formula for Zeta of Weighted Graph

Given weights \( \nu(e) \) on edges. For non-0, variables set \( w_{ab} = \nu(a) \) in \( W \) matrix & get weighted graph zeta. Call matrix \( W_\nu \).

So obtain \( \zeta(u, X, \nu)^{-1} = \det(I - W_\nu) \).

If we make added assumption \( \nu(e^{-1}) = 2 - \nu(e) \), then Bass proof (as in Snowbird volume paper) gives an Ihara-type formula with a new \( A \).

\[
(A_\nu)_{a,b} = \sum_e u^{\nu(e) - 1}_{a \rightarrow b}
\]

It’s old if \( \nu = 1 \).

\[
\zeta(u, X, \nu)^{-1} = (1 - u^2)^{r-1} \det(1 - A_\nu u + Qu^2)
\]
Example. Dumbbell Graph

\[ \zeta_k(W, D)^{-1} = \det \begin{pmatrix}
  w_{aa} -1 & w_{ah} & 0 & 0 & 0 & 0 \\
  0 & -1 & w_{bc} & 0 & 0 & w_{bf} \\
  0 & 0 & w_{cc} -1 & 0 & w_{ce} & 0 \\
  0 & w_{db} & 0 & w_{dd} -1 & 0 & 0 \\
  w_{ea} & 0 & 0 & w_{ed} & -1 & 0 \\
  0 & 0 & 0 & 0 & w_{fe} & w_{ff} -1
\end{pmatrix} \]

Here \( b \) & \( e \) are vertical edges.
Specialize all variables with \( b \) & \( e \) to be 0
get zeta fn of subgraph with vertical edge removed
Fission.

Artin L-Functions of Graphs
Graph $Y$ an unramified covering of Graph $X$ means (assuming no loops or multiple edges)

$\pi : Y \rightarrow X$ is an onto graph map such that

- for every $x \in X$ & for every $y \in \pi^{-1}(x)$,
- $\pi$ maps the points $z \in Y$ adjacent to $y$ 1-1, onto the points $w \in X$ adjacent to $x$.

Normal $d$-sheeted Covering means:

$\exists d$ graph isomorphisms $g_1, \ldots, g_d$ mapping $Y \rightarrow Y$ such that $\pi g_j(y) = \pi(y), \forall y \in Y$

Galois group $G(Y/X) = \{ g_1, \ldots, g_d \}$.

Graph Galois Theory

Gives generalization of Cayley & Schreier graphs ($\alpha, g$)

First pick a spanning tree in $X$ (no cycles, connected, includes all vertices of $X$).

Second make $n = |G|$ copies of the tree $T$ in $X$. These are the sheets of $Y$.
Label the sheets with $g \in G$. Then

- $g(\text{sheet } h) = \text{sheet}(gh)$
- $g(\alpha, h) = (\alpha, gh)$
- $g(\text{path from } (\alpha, h) \text{ to } (\beta, j)) = \text{path from } (\alpha, gh) \text{ to } (\beta, gj)$

How to Label the Sheets of a Covering

Given $G$, get examples $Y$ by giving permutation representation of generators of $G$ to lift edges of $X$ left out of $T$.
Example 1. Quadratic Cover

Cube covers Tetrahedron

Spanning Tree in X is red.
Corresponding sheets of Y are also red

Example of Splitting of Primes in Quadratic Cover

Picture of Splitting of Prime which is inert;
i.e., f=2, g=1, e=1
1 prime cycle D above, & D is lift of C².
Example of Splitting of Primes in Quadratic Cover

Picture of Splitting of Prime which splits completely; i.e., \( f=1, g=2, e=1 \)
2 primes cycles above

Frobenius Automorphism

\[
\text{Frob}(D) = \left( \frac{Y/X}{D} \right) = ji^{-1} \in G = \text{Gal}(Y/X)
\]
where \( ji^{-1} \) maps sheet \( i \) to sheet \( j \)

The unique lift of \( C \) in \( Y \) starts at \( (\alpha, i) \) ends at \( (\alpha, j) \)

Exercise: Compute \( \text{Frob}(D) \) on preceding pages, \( G = \{1, g\} \).
Properties of Frobenius

1) Replace \((\alpha, i)\) with \((\alpha, hi)\). Then \(\text{Frob}(D) = j_i^{-1}\) is replaced with \(h_j i^{-1} h^{-1}\). Or replace \(D\) with different prime above \(C\) and see that

Conjugacy class of \(\text{Frob}(D) \in \text{Gal}(Y/X)\) unchanged.

2) Varying \(\alpha = \text{start of } C\) does not change \(\text{Frob}(D)\).

3) \(\text{Frob}(D)^j = \text{Frob}(D^j)\).

Artin L-Function

\(\rho = \text{representation of } G = \text{Gal}(Y/X), \ u \text{ complex, } |u| \text{ small}\)

\[
L(u, \rho, Y/X) = \prod_{[C]} \det \left( 1 - \rho \left( \frac{Y/X}{D} \right) u^{\nu(C)} \right)^{-1}
\]

\([C] = \text{primes of } X\)

\(\nu(C) = \text{length } C, \ D \text{ a prime in } Y \text{ over } C\)

Properties of Artin L-Functions

1) \(L(u, 1, Y/X) = \zeta(u, X) = \text{Ihara zeta function of } X\)

(our analogue of the Dedekind zeta function, also Selberg zeta)

2) \[
\zeta(u, Y) = \prod_{\rho} L(u, \rho, Y/X)^{d_{\rho}}
\]

product over all irreducible reps of \(G, \ d_{\rho} = \text{degree } \rho\)
Edge Artin L-Function

Defined as before with edge norm and representation $\rho$

$$L_E(W, \rho, Y/X) = \prod_{C} \det(I - \rho([Y/X, D]) N_E(C))^{-1}$$

Let $m = |E|$. Define $W_\rho$ to be a $2dm \times 2dm$ matrix with $e, f$ block given by $w_{ef} \rho(\sigma(e))$. Then

$$L_E(W_\rho, Y/X) = \det(I - W_\rho)^{-1}.$$ 

Ihara Theorem for L-Functions

$$L(u, \chi_\rho, Y/X)^{-1} = (1 - u^2)^{(r-1)d_\rho} \det(I' - A'_\rho u + Q' u^2)$$

$r =$ rank fundamental group of $X = |E| - |V| + 1$
$\rho =$ representation of $G = \text{Gal}(Y/X)$, $d = d_\rho =$ degree $\rho$

Definitions. $n \times n$ matrices $A', Q', I'$, $n = |X|

nxn matrix $A(g)$, $g \in \text{Gal}(Y/X)$, has entry for $\alpha, \beta \in X$ given by

$$(A(g))_{\alpha, \beta} = \# \{ \text{edges in } Y \text{ from } (\alpha, e) \text{ to } (\beta, g) \},$$

$$A'_\rho = \sum_{g \in G} A(g) \otimes \rho(g)$$

$Q = \text{diagonal matrix}, \ j\text{th diagonal entry} = q_j$

$Q' = Q \otimes I_d$, $I' = I_{nd} = \text{identity matrix.}$
Y = cube, X = tetrahedron: \( G = \{e, g\} \)

representations of \( G \) are 1 and \( \rho \):
\[ \rho(e) = 1, \quad \rho(g) = -1 \]

\[ A(e)_{u,v} = \# \{ \text{length 1 paths } u' \text{ to } v' \text{ in } Y \} \]

\[ A(g)_{u,v} = \# \{ \text{length 1 paths } u' \text{ to } v'' \text{ in } Y \} \]

\[
A(e) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\quad A(g) = \begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix}
\]

\[ A'_{1} = A = \text{adjacency matrix of } X = A(e) + A(g) \]

\[
A'_{\rho} = A(e) - A(g) = \begin{pmatrix}
0 & 1 & -1 & -1 \\
1 & 0 & 1 & 1 \\
-1 & 1 & 0 & -1 \\
-1 & 1 & -1 & 0
\end{pmatrix}
\]

**EXAMPLE**

\[ \zeta(u, Y)^{-1} = L(u, \rho, Y/X)^{-1} \zeta(u, X)^{-1} \]

\[ L(u, \rho, Y/X)^{-1} = (1-u^2) (1+u) (1+2u) (1-u+2u^2)^3 \]

\[ \zeta(u, X)^{-1} = (1-u^2)(1-u)(1-2u) (1+u+2u^2)^3 \]
Examples of Pole Distribution for Covers of Small Irregular Unweighted Graph
Poles of Ihara Zeta of $\mathbb{Z}_{10001}$ Cover of 2 Loops + Extra Vertex are pink dots.

Circles Centers $(0,0)$; Radii: $3^{-1/2}, R^{1/2}, 1$; $R \approx 0.4694$

$\mathbb{Z}_m \times \mathbb{Z}_n$ cover of 2-Loops Plus Vertex

Sheets of Cover indexed by $(x,y)$ in $\mathbb{Z}_m \times \mathbb{Z}_n$

The edge $L$-fns for Characters

$\chi_{r,s}(x,y) = \exp[2\pi i ((rx/m) + (sy/n))]$

Normalized Frobenius $(a) = (1,0)$

Normalized Frobenius $(b) = (0,1)$

The picture shows $m=n=3$. 
Poles of Ihara Zeta for a $Z_{101} \times Z_{163}$-Cover of 2 Loops + Extra Vertex are pink dots

Circles Centers (0,0): Radii: $3^{-1/2}$, $R^{1/2}$, 1; $R \approx .47$

$Z$ is random 407 cover of 2 loops plus vertex graph in picture. The pink dots are at poles of $\zeta_Z$.

Circles have radii $q^{-1/2}$, $R^{1/2}$, $p^{-1/2}$, with $q=3$, $p=1$, $R \approx .4694$
1) Find the meaning of the Riemann hypothesis for irregular graphs. Are there functional equations? How does it compare with Lubotzky’s definition of Ramanujan irregular graph?

2) For regular graphs, can you define a W-matrix to make the spacings of poles of zetas that look Poisson become GOE?

3) For a large Galois cover of a fixed base graph, can you produce a distribution of poles that looks like that of a random cover?

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**Homework Problems**

1) Find the meaning of the Riemann hypothesis for irregular graphs. Are there functional equations? How does it compare with Lubotzky’s definition of Ramanujan irregular graph?

2) For regular graphs, can you define a W-matrix to make the spacings of poles of zetas that look Poisson become GOE?

3) For a large Galois cover of a fixed base graph, can you produce a distribution of poles that looks like that of a random cover?
References: 3 papers with Harold Stark in Advances in Math.
Paper with Matthew Horton & Harold Stark in Snowbird Proceedings
See my website for draft of a book:

www.math.ucsd.edu/~aterras/newbook.pdf