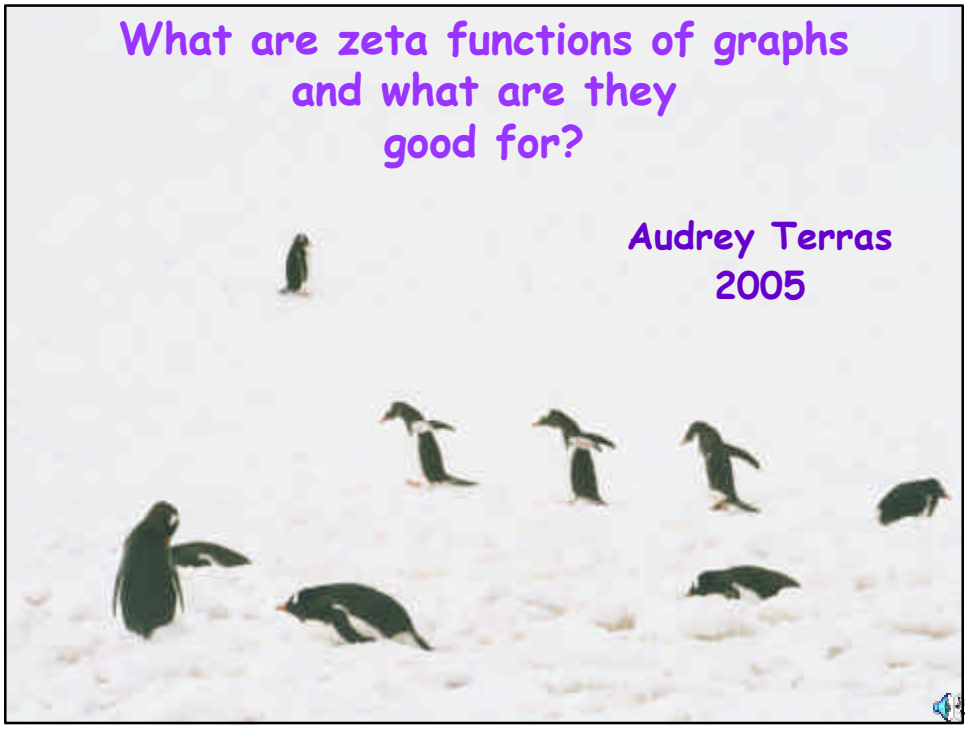
What are zeta functions of graphs and what are they good for?

Audrey Terras
2005



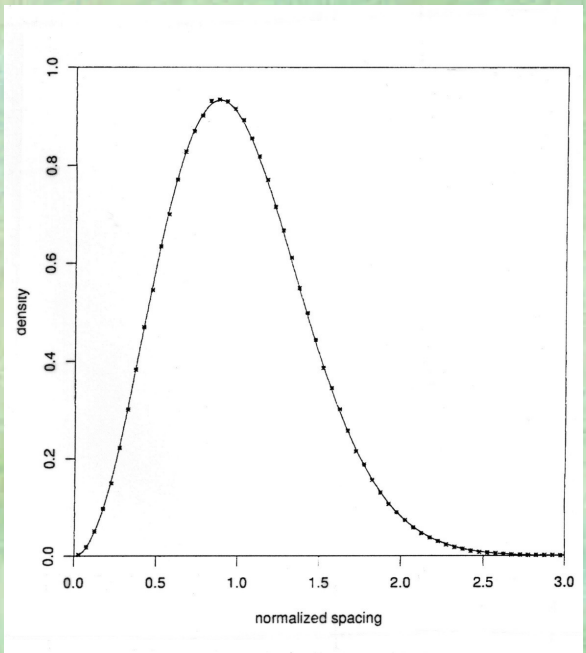
Introduction

The Riemann zeta function for Re(s)>1

$$z(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=prime} \left(1 - p^{-s}\right)^{-1}.$$

- Riemann extended to all complex s with pole at s=1.
- Functional equation relates value at s and 1-s
- Riemann hypothesis
- Duality between primes and complex zeros of zeta
- See Davenport, Multiplicative Number Theory.

Odlyzko's Comparison of Spacings of Imaginary Parts of Zeros of Zeta and Eigenvalues of Random Hermitian Matrix.



Many Kinds of Zeta

Dedekind zeta of an algebraic number field F, where primes become prime ideals $\mathfrak{p}$ and infinite product of terms

$(1-N\mathfrak{p}^{-s})^{-1}$, where $N\mathfrak{p}$ = norm of $\mathfrak{p}$ = $\#(\mathfrak{O}/\mathfrak{p})$,

$\mathfrak{O}$=ring of integers in F

Selberg zeta associated to a compact Riemannian manifold M=G\H

H = upper half plane with $ds^2=(dx^2+dy^2)y^{-2}$

G=discrete subgroup of group of real Möbius transformations

primes = primitive closed geodesics C in M of length n(C)

(primitive means only go around once)

$$Z(s)=\prod_{[C]}\prod_{j\geq 0}\left(1-e^{-(s+j)n(C)}\right)$$

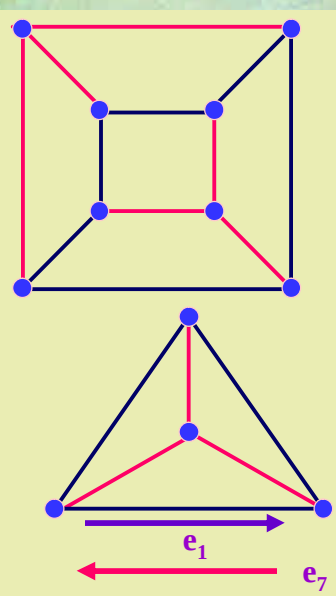
Duality between spectrum D on M & lengths closed geodesics in M
Z(s+1)/Z(s) is more like Riemann zeta

Zeta & L-Functions of Graphs

We will see they have similar properties and applications to those of number theory

But first we need to figure out what primes in graphs are

Labeling Edges of Graphs



X = finite connected (not-necessarily regular graph)

Orient the edges.

Label them as follows.

Here the inverse edge has opposite orientation.

$$e_1, e_2, \ldots, e_{|E|},$$

$$e_{|E|+1} = e_1^{-1}, \ldots, e_{2|E|} = e_{|E|}^{-1}$$

We will use this labeling in the next section on edge zetas

"PRIMES in GRAPHS"

are equivalence classes of closed backtrackless tailless primitive paths

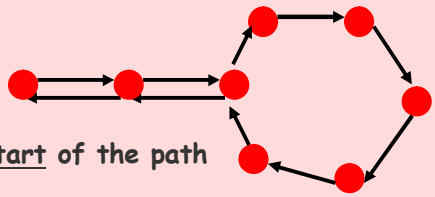
DEFINITIONS

backtrack

equivalence class: change starting point

tail

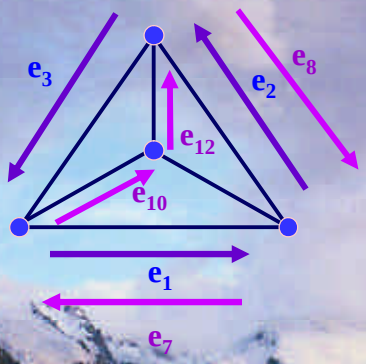
a



Here a is the <u>start</u> of the path

non-primitive: go around path more than once

EXAMPLES of Primes in a Graph



$[C] = [e_1 e_2 e_3]$

$[C'] = [e_7 e_{10} e_{12} e_8]$

Ihara Zeta Function

$$z_V(u,X) = \prod_{\substack{[C] \\ prime}} \left(1 - u^{n(C)}\right)^{-1}$$

Ihara's Theorem (Bass, Hashimoto, etc.)

A = adjacency matrix of X

Q = diagonal matrix; jth diagonal entry = degree jth vertex -1;

r = rank fundamental group = |E|-|V|+1

$$z_V(u,X)^{-1} = (1-u^2)^{r-1} \det(I - Au + Qu^2)$$

Here V is for vertex

Some History 1960-present

Ihara defined the zeta as a product over p-adic group elements.

Serre saw the graph theory interpretation.

Sunada, Hashimoto, Bass, etc. extended the theory.

This is intended to be an introduction to Stark and Terras, Advances in Math, 1996, 2000

Remarks for q+1-Regular Graphs Mostly

- k(X)=the number of spanning trees of X, the complexity

$$\left(\frac{1}{z_X}\right)^{(r)}(1) = r!(-1)^{r+1}2^r(r-1)k(X)$$

  analogue of value of Dedekind zeta at 0
- Riemann Hypothesis, (poles), means graph is Ramanujan i.e., non-trivial spectrum of adjacency matrix is contained in the interval (-2Öq, 2Öq) = spectrum for the universal covering tree [see Lubotzky, Phillips & Sarnak, Combinatorica, 8 (1988)]. Here $u=q^{-s}$.
- Ihara zeta has functional equations relating value at u and 1/(qu), q=degree - 1

- Alon conjecture RH is true for "most" graphs but can be false
- Hashimoto [Adv. Stud. Pure Math., 15 (1989)] proves Ihara z for certain graphs is essentially the z function of a Shimura curve over a finite field
- The Prime Number Theorem Let $p_X(m)$ denote the number of primes [C] in X with length m.

  Assume X finite connected (q+1)-regular not bipartite.

  1/q = absolute value of closest pole(s) of z(u,X) to 0, implies

$$p_X(m) \sim q^m/m \quad \text{as } m \to \infty.$$

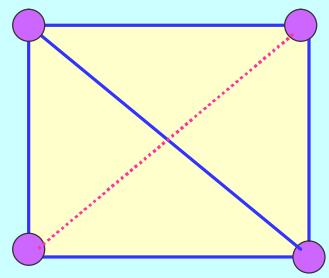
  The proof comes from exact formula for $p_X(m)$ by analogous method to that of Rosen, Number Theory in Function Fields, page 56. N_m=# closed paths length m no backtrack, no tails

$$u\frac{d\log z(u,X)}{du} = \sum_{m=1}^{\infty} N_m u^m$$

  For irregular graph q is replaced by 1/R,

  R=radius of convergence of z

2 Examples K4 and X=K4-edge

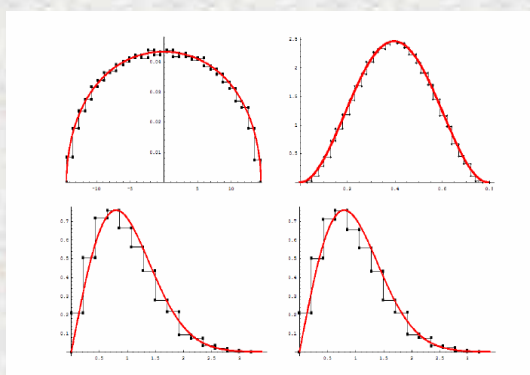


$$z_V(u,K_4)^{-1} = (1-u^2)^2(1-u)(1-2u)(1+u+2u^2)^3$$

$$z_V(u,X)^{-1} = (1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3)$$

Derek Newland's Experiments

Mathematica experiment with random 53-regular graph - 2000 vertices



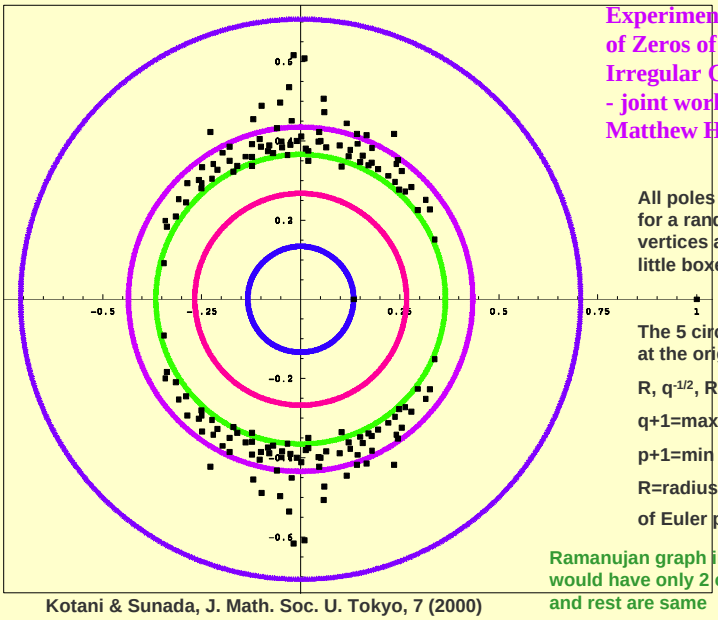
Spectrum adjacency matrix

$z(q^{-s})$ *as function of* s

Top row = distributions for eigenvalues of A on left and Imaginary parts of the zeta poles on right.

Bottom row contains their respective level spacings.

Red line on bottom: Wigner surmise, $y = (\pi x/2)\exp(-\pi x^2/4)$.



Experiments on Locations of Zeros of Ihara Zeta of Irregular Graphs - joint work with Matthew Horton

All poles except -1 of $Z_X(u)$ for a random graph with 80 vertices are denoted by little boxes.

The 5 circles are centered at the origin and have radii

R, $q^{-1/2}$, $R^{1/2}$, $(pq)^{-1/4}$, $p^{-1/2}$

$q+1$=max degree,

$p+1$=min degree

R=radius of convergence of Euler product for $Z_X(u)$

Ramanujan graph in regular case would have only 2 circles; inner and rest are same

All poles but $\pm q$ on green circle; radius $\sqrt{q}$

Kotani & Sunada, J. Math. Soc. U. Tokyo, 7 (2000) show imaginary poles lie between pink and outside circles; all poles between inner circle and circle of radius 1

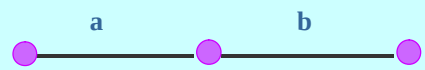
Edge Zetas

Orient the edges of the graph. Recall the labeling!

Define Edge matrix W to have a,b entry w_{ab} in $\mathbb{C}$ & set

$w(a,b)=w_{ab}$

if the edges a and b look like those below and $a^{1}b^{-1}$



Otherwise set $w_{ab} = 0$

W is $2|E| \times 2|E|$ matrix

If $C = a_1 a_2 \frac{1}{4} a_s$ where a_j is an edge, define edge norm to be

$$N_E(C) = w(a_s, a_1) w(a_1, a_2) w(a_2, a_3) \cdots w(a_{s-1}, a_s)$$

Edge Zeta

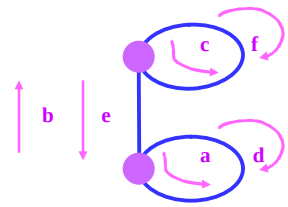
$$z_E(W, X) = \prod_{\substack{[C] \\ prime}} \left(1 - N_E(C)\right)^{-1}$$

Properties of Edge Zeta

- Set all non-0 variables $w_{ab}=u$ in the edge zeta & get Ihara zeta
- If you cut an edge of a graph, compute the edge zeta by setting all variables equal to 0 if the cut edge or its inverse appear in subscripts
- Edge zeta is the reciprocal of a polynomial given by a much simpler determinant formula than the Ihara zeta
- Even better, the proof is simpler (compare Bowen & Lanford proof for dynamical zetas)
- and Bass deduces Ihara from this

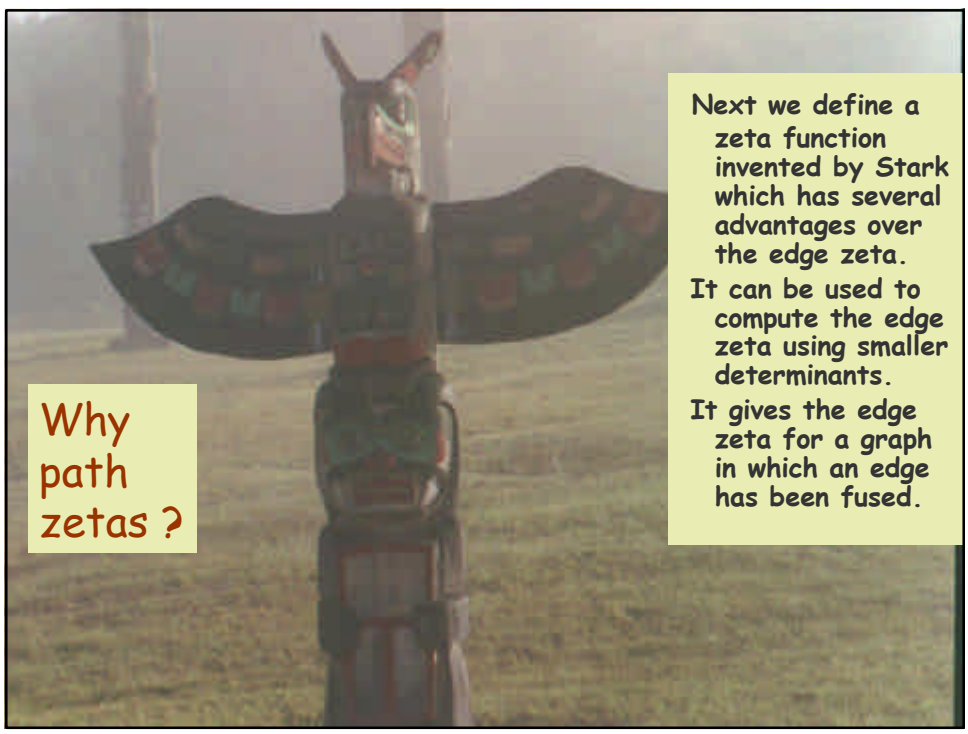
$$z_E(W, X) = \det(I - W)^{-1}$$

Example. Dumbbell Graph



$$z_E(W,D)^{-1} = \det \begin{pmatrix} w_{aa}-1 & w_{ab} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{bc} & 0 & 0 & w_{bf} \\ 0 & 0 & w_{cc}-1 & 0 & w_{ce} & 0 \\ 0 & w_{db} & 0 & w_{dd}-1 & 0 & 0 \\ w_{ea} & 0 & 0 & w_{ed} & -1 & 0 \\ 0 & 0 & 0 & 0 & w_{fe} & w_{ff}-1 \end{pmatrix}$$

Here b & e are vertical edges.
Specialize all variables with b & e to be 0
get zeta fn of subgraph with vertical edge removed
Fission.



Path Zeta Function

Fundamental Group of X can be identified with group generated by edges left out of a spanning tree

$$e_1, \ldots e_r, e_1^{-1}, \ldots, e_r^{-1}$$

Define 2r ×2r **path matrix** Z - ij entry

z_{ij} in $\mathbb{C}$ if $e_j \neq e_i^{-1}$ and z_{ij} =0, otherwise.

Imitate definition of edge zeta function. Define for prime path

$C = a_1 \cdots a_s,\ where\ a_j \in \{e_1^{\pm 1}, \ldots, e_r^{\pm 1}\}$

path norm

$$N_P(C) = z(a_s, a_1) \prod_{i=1}^{s-1} z(a_i, a_{i+1})$$

Define **path zeta**

$$z_P(Z, X) = \prod_{\substack{[C] \\ prime}} \left(1 - N_P(C)\right)^{-1}$$

Specialize Path Zeta to Edge Zeta

edges left out of a spanning tree T of X are $e_1, \ldots e_r$

inverse edges are $e_{r+1} = e_1^{-1}, \ldots, e_{2r} = e_r^{-1}$

edges of the spanning tree T are $t_1, \ldots, t_{|X|-1}$

with inverse edges $t_{|X|}, \ldots t_{2|X|-2}$

A prime cycle *C* is first written as a product of the generators of the fundamental group e_j and then a product of actual edges e_j and t_k.

Do this by inserting $t_{k_1} \cdots t_{k_n}$

which is unique path on T joining end vertex of e_i & start vertex of e_j if e_i and e_j are adjacent in *C*.

Now **specialize the path matrix Z to Z(W)** with entries

$$z_{ij} = w(e_i, t_{k_1}) w(t_{k_n}, e_j) \prod_{n=1}^{n-1} w(t_{k_n}, t_{k_{n+1}})$$

Then

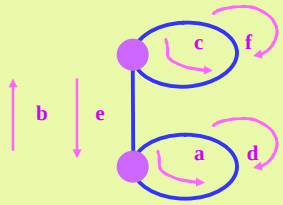
$$z_P(Z(W), X) = z_E(W, X)$$

Example – Again the Dumbbell

Recall that the edge zeta involved a 6x6 determinant.

The path zeta is only 4x4.

Maple computes it much faster than the 6x6.



e.g., specialize z_{ac} to $w_{ab}w_{bc}$

Fusion:

shrink edge b to a point.

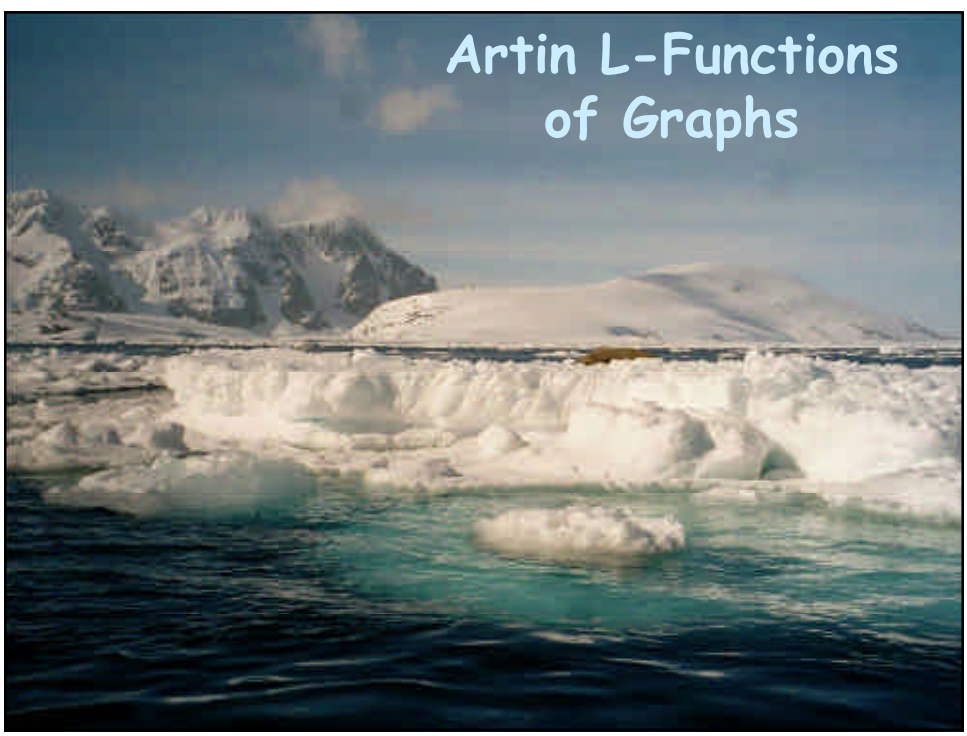
edge zeta of new graph obtained by setting

$$w_{xb}w_{by}=w_{xy}$$

in specialized path zeta &

same for e instead of b.

$$z_E(W,X)^{-1} = \det\begin{pmatrix} w_{aa}-1 & w_{ab}w_{bc} & 0 & w_{ab}w_{bf} \\ w_{ce}w_{ea} & w_{cc}-1 & w_{ce}w_{ed} & 0 \\ 0 & w_{db}w_{bc} & w_{dd}-1 & w_{db}w_{bf} \\ w_{fe}w_{ea} & 0 & w_{fe}w_{ed} & w_{ff}-1 \end{pmatrix}$$



Graph Galois Theory

Gives generalization of Cayley & Schreier graphs

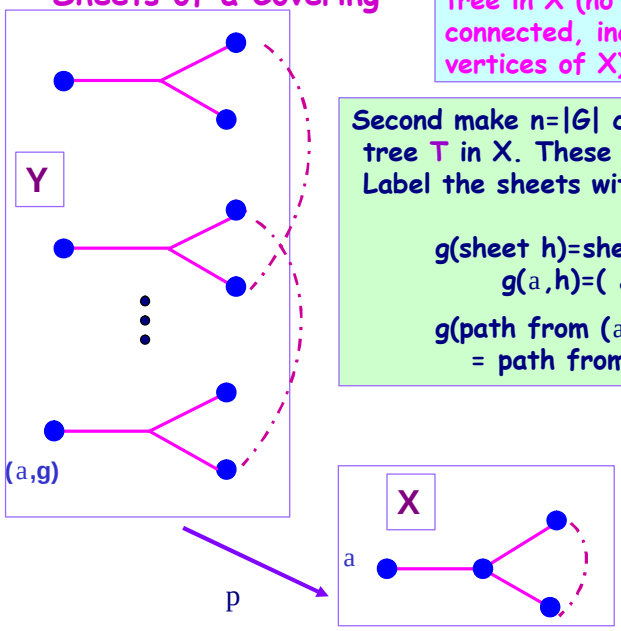
Graph Y an unramified covering of Graph X means
(assuming no loops or multiple edges)

$p: Y \to X$ is an onto graph map such that
for every $x \in X$ & for every $y \in p^{-1}(x)$,
p maps the points $z \in Y$ adjacent to y
1-1, onto the points $w \in X$ adjacent to x.

Normal d-sheeted Covering means:
$\exists$ d graph isomorphisms $g_1, \ldots, g_d$ mapping $Y \to Y$
such that $p\, g_j(y) = p(y)$ " $y \in Y$
Galois group $G(Y/X) = \{ g_1, \ldots, g_d \}$.

How to Label the Sheets of a Covering

First pick a spanning tree in X (no cycles, connected, includes all vertices of X).



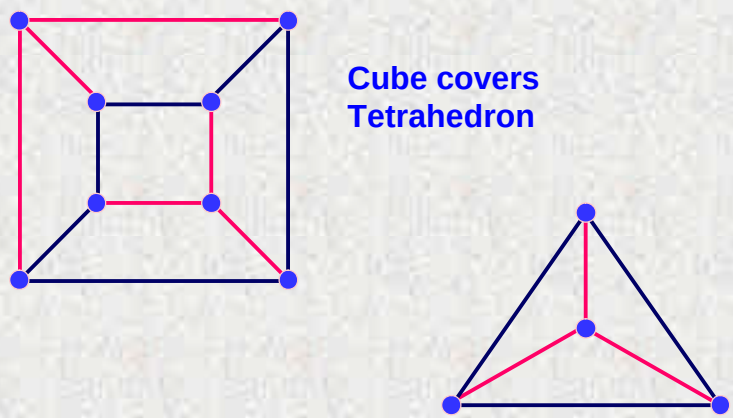
Second make $n=|G|$ copies of the tree T in X. These are the sheets of Y. Label the sheets with $g \in G$. Then

$g(\text{sheet } h) = \text{sheet}(gh)$
$g(a,h) = (a, gh)$

$g(\text{path from } (a,h) \text{ to } (b,j))$
$= \text{path from } (a,gh) \text{ to } (b,gj)$

Given G, get examples Y by giving permutation representation of generators of G to lift edges of X left out of T.

Example 1. Quadratic Cover

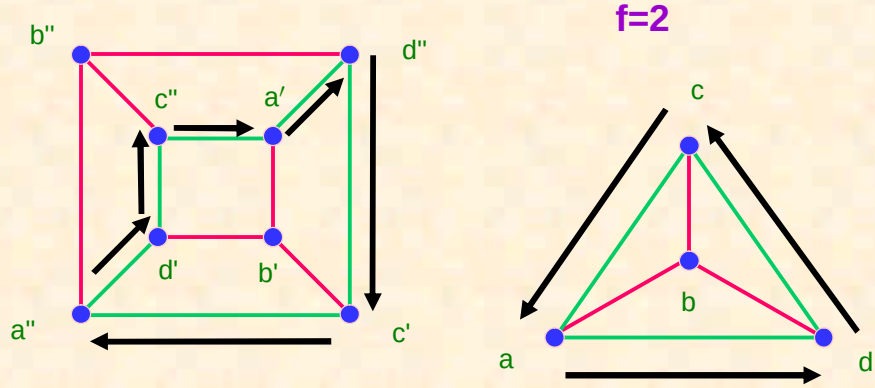


Cube covers Tetrahedron

Spanning Tree in X is red.
Corresponding sheets of Y are also red

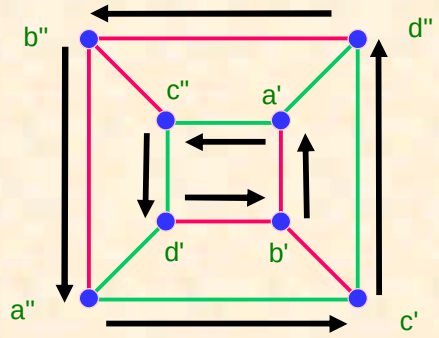
Example of Splitting of Primes in Quadratic Cover

f=2

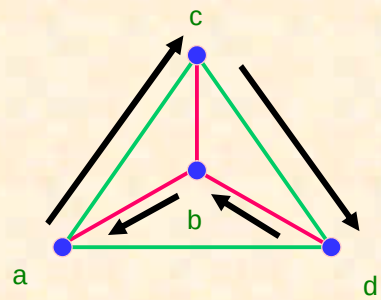


Picture of Splitting of Prime which is inert;
i.e., f=2, g=1, e=1
1 prime cycle D above, & D is lift of C^2.

Example of Splitting of Primes in Quadratic Cover



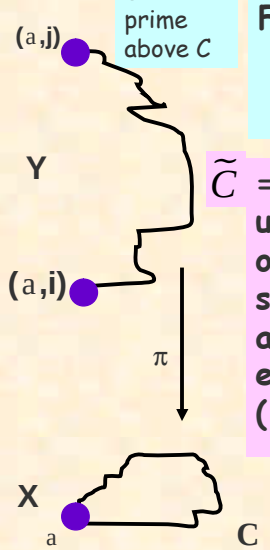
g=2



Picture of Splitting of Prime which splits completely; i.e., **f=1, g=2, e=1**
2 primes cycles above

Frobenius Automorphism

D a prime above C



$\mathrm{Frob}(D) = \left(\frac{Y/X}{D}\right) = ji^{-1} \in G = \mathrm{Gal}(Y/X)$

where ji^{-1} maps sheet i to sheet j

$\widetilde{C}$ = the unique lift of C in Y starting at (a,i) ending at (a,j)

$\widetilde{C}$ not necessarily closed

$length(\widetilde{C}) = length(C)$

(D a prime above C is closed and is obtained by f liftings like $\widetilde{C}$)

Exercise: Compute Frob(D) on preceding pages, G={1,g}.

Properties of Frobenius

1) Replace (a,i) with (a,hi). Then Frob(D) = ji^{-1} is replaced with $hji^{-1}h^{-1}$. Or replace D with different prime above C and see that
   Conjugacy class of Frob(D) Î Gal(Y/X) unchanged.
2) Varying a=start of C does not change Frob(D).
3) $\mathrm{Frob}(D)^j = \mathrm{Frob}(D^j)$.

Artin L-Function

r = representation of G=Gal(Y/X), u Î ℂ, |u| small

$$L(u, r, Y/X) = \prod_{[C]} \det\left(1 - r\left(\frac{Y/X}{D}\right)u^{n(C)}\right)^{-1}$$

[C]=primes of X
n(C)=length C, D a prime in Y over C

Properties of Artin L-Functions

Copy from Lang, Algebraic Number Theory

1) L(u,1,Y/X) = z(u,X) = **Ihara zeta function** of X (our analogue of the **Dedekind zeta** function, also **Selberg zeta)**
2)
$$z(u, Y) = \prod_{r \in \widehat{G}} L(u, r, Y/X)^{d_r}$$

product over all irreducible reps of G, d_r=degree r

Proofs of 1) and 2) require basic facts about reps of finite groups. See A. T., Fourier Analysis on Finite Groups and Applications.

Ihara Theorem for L-Functions

$$L(u, c_r, Y/X)^{-1}$$
$$= (1-u^2)^{(r-1)d_r} \det(I' - A'_r u + Q' u^2)$$

r=rank fundamental group of X = |E|-|V|+1
r= representation of G = Gal(Y/X), d = d_r = degree r

Definitions. nd´nd matrices A′, Q′, I′, n=|X|

nxn matrix A(g), g Î Gal(Y/X), has entry for a,b ÎX given by $(A(g))_{a,b}$ = # { edges in Y from (a,e) to (b,g) }, e=identity Î G.

$$A'_r = \sum_{g \in G} A(g) \otimes r(g)$$

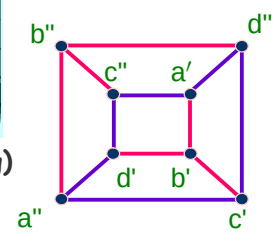
Q = diagonal matrix,
jth diagonal entry = q_j = (degree of jth vertex in X)-1,
Q′ = QÄI_d , I′ = I_{nd} = identity matrix.

EXAMPLE

Y=cube, X=tetrahedron: G = {e,g}
representations of G are 1 and r: r(e) = 1, r(g) = -1
$A(e)_{u,v}$ = #{ length 1 paths u′ to v′ in Y}
$A(g)_{u,v}$ = #{ length 1 paths u′ to v″ in Y}

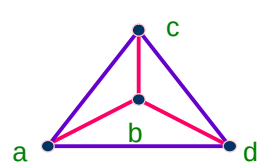
(u,e)=u'
(u,g)=u''

$$A(e) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad A(g) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$



A'_1 = A = adjacency matrix of X = A(e)+A(g)

$$A'_r = A(e) - A(g) = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{pmatrix}$$

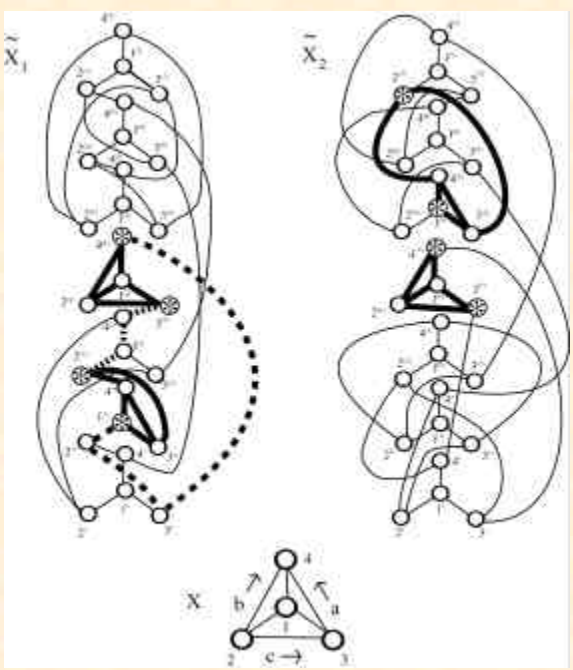


Zeta and L-Functions of Cube & Tetrahedron

- $L(u, r, Y/X)^{-1} = (1-u^2)(1+u)(1+2u)(1-u+2u^2)^3$
- $z(u,Y)^{-1} = L(u, r, Y/X)^{-1}\, z(u,X)^{-1}$
- $z(u,X)^{-1} = (1-u^2)^2(1-u)(1-2u)(1+u+2u^2)^3$
- poles of $z(u,X)$ are $\{1,1,1,-1,-1,\frac{1}{2},r,r,r\}$
  where $r=(-1\pm\ddot{0}-7)/4$ and $|r|=1/\ddot{0}2$
  $\frac{1}{2}$=Pole of $z(u,X)$ closest to 0
  governs prime number thm
- Coefficients of generating function below =
  # length n closed paths no backtracking or tails

$$u\frac{d}{du}\log z\,(u,X)$$
$$= 24u^3 + 24u^4 + 96u^6 + 168u^7 + 168u^8$$
$$+528u^9 + 1200u^{10} + 1848u^{11} + O(u^{12})$$

So there are 8 primes of length 3 in X, for example.



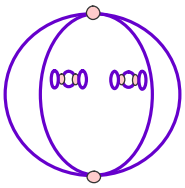
Application of Galois Theory of Graph Coverings. You can't hear the shape of a graph.

2 connected regular graphs (without loops & multiple edges) which are isospectral but not isomorphic

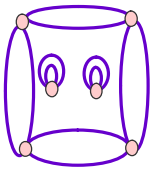
⌘ See A.T. & Stark in Adv. in Math., Vol. 154 (2000) for the details.

⌘ Method goes back to algebraic number theorists who found number fields K_i which are non isomorphic but have the same Dedekind zeta. See Perlis, J. Number Theory, 9 (1977).

⌘ Galois group is $GL(3,\mathbb{F}_2)$, order 168, appears in Buser, Geometry & Spectra of Compact Riemann Surfaces also Gordon, Webb & Wolpert (isospectral non-isomorphic planar drums) Inventiones, 110 (1992).



Audrey



Harold

⌘ Robert Perlis and Aubi Mellein have used the same methods to find many examples of isospectral non isomorphic graphs with multiple edges and components. 2 such are on the right.

Homework Problems

1) What is the zeta function of a quantum or weighted graph?
2) Find the meaning of the Riemann hypothesis for irregular graphs. Are there functional equations?
3) Are there analogs of Artin L-functions for higher dimensional things – buildings ?
4) Connect the zeta polynomials of graphs to other polynomials associated to graphs and knots (Tutte, Alexander, and Jones polynomials)
5) Is there a graph analog of regulator, Stark Conjectures, class field theory for abelian graph coverings? Or more simply a quadratic reciprocity law, fundamental units? The ideal class group is the Jacobian of a graph and has order = number of spanning trees (paper of Roland Bacher, Pierre de la Harpe and Tatiana Nagnibeda). There is an analog of Brauer-Siegel theory (see H.S. and A.T. , Part III).