Math 202B HW3 (Selected solutions)

1) Let $\rho_1, \rho_2 : G \to GL_n(\mathbb{C})$, such that $\forall g \in G$, $\exists A_g \in GL_n(\mathbb{C})$ so that $A_g \rho_1(g) A_g^{-1} = \rho_2(g)$.

$$\text{Then } \chi_{\rho_1}(g) = \operatorname{tr} \rho_1(g) \underset{\uparrow}{=} \operatorname{tr}\left(A_g \rho_1(g) A_g^{-1}\right) = \operatorname{tr} \rho_2(g) = \chi_{\rho_2}(g).$$

because $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

Thus $\chi_{\rho_1} = \chi_{\rho_2}$, so $(\rho_1, \mathbb{C}^n) \cong (\rho_2, \mathbb{C}^n)$, which means there is some $A \in GL_n(\mathbb{C})$ so that $A\rho_1(g)A^{-1} = \rho_2(g)$ for all $g \in G$.

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3) i) if $[G:H] = 2$, then H has two left- and right-cosets, each represented by any $g \notin H$. So $G = H \sqcup gH = H \sqcup Hg$. Thus, for all $g \notin H$, $gH = Hg$. Of course $hH = H = Hh$ for all $h \in H$, so we see H is normal in G.

So the quotient map $G \xrightarrow{\varepsilon} G/H$ is a homomorphism with $\ker \pi = H$

Since $|G/H| = [G:H] = 2$, we see $G/H \cong \{\pm 1\}$ is the only group of order 2.
So $\varepsilon : G \to \{\pm 1\}$ has $\ker \varepsilon = H$.

ii & iii) Let (ρ, V) be an irreducible representation of G. Consider $(\rho|_H, V)$, a representation of H.

$$1 = \langle \chi_V, \chi_V \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\overline{\chi_V(g)}$$

$$1 = \frac{1}{|G|} \sum_{h \in H} \chi_V(h)\overline{\chi_V(h)} + \frac{1}{|G|} \sum_{g \notin H} \chi_V(g)\overline{\chi_V(g)}$$

$$1 = \underbrace{\frac{|H|}{|G|}}_{\frac{1}{2}} \langle \chi_V|_H, \chi_V|_H \rangle_H + \frac{1}{|G|} \sum_{g \notin H} \chi_V(g)\overline{\chi_V(g)}$$

$$\text{So } \langle \chi_V|_H, \chi_V|_H \rangle_H = 2 - \frac{2}{|G|} \sum_{g \notin H} |\chi_V(g)|^2$$

Since $\langle \chi_V|_H, \chi_V|_H \rangle_H$ is a positive integer, and $\frac{2}{|G|} \sum_{g \notin H} |\chi_V(g)|^2 \geq 0$,

we see $\langle \chi_V|_H, \chi_V|_H \rangle_H = \begin{cases} 2 & \text{if } \chi_V(g) = 0 \text{ for all } g \notin H \\ 1 & \text{otherwise} \end{cases}$

So $(\rho|_H, V)$ is a reducible representation of $H \Leftrightarrow \langle \chi_V|_H, \chi_V|_H \rangle_H = 2$

$$\Leftrightarrow \forall g \notin H,\ \chi_V(g) = 0$$

$$\Leftrightarrow \forall g \notin H,\ \chi_V(g) = -\chi_V(g) = (\varepsilon \cdot \chi_V)(g)$$

$$\Leftrightarrow \rho \cong \varepsilon \cdot \rho$$

In this case,

Since $\langle \chi_V|_H, \chi_V|_H \rangle_H = 2 = 1^2 + 1^2$, we see V decomposes into two nonisomorphic subrepresentations as $V_1 \oplus V_2$

So $1 = \langle \chi_V|_H, \chi_{V_1} \rangle_H = \langle \chi_V, \mathrm{Ind}_H^G \chi_{V_1} \rangle_G$, meaning $V \subseteq \mathrm{Ind}_H^G V_1$ since V is irreducible.

By taking dimensions we see $\dim V \leq \dim \mathrm{Ind}_H^G V_1 = [G:H] \dim V_1 = 2 \dim V_1$,

meaning $\dim V_1 \geq \frac{1}{2} \dim V$.

Since V_2 plays the same role as V_1, we see $\dim V_2 \geq \frac{1}{2} \dim V$ as well,

and $\dim V = \dim V_1 + \dim V_2 \geq \frac{1}{2} \dim V + \frac{1}{2} \dim V = \dim V$, so both inequalities are in fact equalities. Thus $\dim V_1 = \dim V_2 = \frac{1}{2} \dim V$.

Finally, since $V \subseteq \mathrm{Ind}_H^G V_1$ and $\dim V = \dim \mathrm{Ind}_H^G V_1$, we must have $V \cong \mathrm{Ind}_H^G V_1$, and likewise $V \cong \mathrm{Ind}_H^G V_2$.

6) $D_{2n} = \langle \sigma, c : c^n = 1 = \sigma^2, \sigma c \sigma = c^{-1} \rangle$

Let $C = \langle c \rangle$ be the cyclic group of order n.

Know all characters of C have the form $\chi_k(c^j) = \omega^{jk}$ where $\omega = e^{2\pi i/n}$ is an n^{th} root of unity

Let V_k be the vector space corresponding to χ_k

Then $\tilde{V}_k = \mathrm{Ind}_C^{D_{2n}} V_k = V_k \oplus \sigma V_k = \mathrm{span}\{e_1^{(k)}, \sigma e_2^{(k)}\}$

$$* \begin{cases} \text{So } c^j \cdot e_1^{(k)} = \omega^{jk} e_1^{(k)}, \quad c^j \cdot \sigma e_2^{(k)} = \sigma c^{-j} e_2^{(k)} = \omega^{-jk} \sigma e_2^{(k)} \\ (c^j\sigma) \cdot e_1^{(k)} = c^j(\sigma e_1^{(k)}) = \omega^{-jk} \sigma e_2^{(k)} \text{ (as above)}, \ (c^j\sigma)(\sigma e_2^{(k)}) = c^j e_2^{(k)} = \omega^{jk} e_2^{(k)} \end{cases}$$

Thus we get $\chi_{\tilde{V}_k}(c^j) = \omega^{jk} + \omega^{-jk} = \omega^{jk} + \overline{\omega^{jk}} = 2\mathrm{Re}(\omega^{jk})$

$\chi_{\tilde{V}_k}(c^j\sigma) = 0$.

Thus we get distinct characters for $0 \le k \le \frac{n}{2}$ — $\tilde{V}_k \cong \tilde{V}_{n-k}$ from the above.

$\langle \tilde{V}_0, \tilde{V}_0 \rangle = 2$, so problem 5 shows $\tilde{V}_0 \cong W_1 \oplus \varepsilon \cdot W_1$. Checking this, we see $W_1 =$ the trivial rep.

- For n odd, computing inner products confirms $\tilde{V}_k$ is irreducible for $k = 1, 2, \dots, \frac{n-1}{2}$

  Thus we have $\frac{n-1}{2}$ 2-dimensional irreps and 2 1-dimensional irreps.

  Since $1^2 + 1^2 + \frac{n-1}{2}(2^2) = 2 + 2(n-1) = 2n$, these are all.

- For n even, we get $\tilde{V}_1, \dots, \tilde{V}_{\frac{n-2}{2}}$ irreducible and $\tilde{V}_{\frac{n}{2}}$ reducible. Thus $\tilde{V}_{\frac{n}{2}} \cong U \oplus \varepsilon \cdot U$,

  where we can check $\chi_U(c^j\sigma^k) = (-1)^j$, $\chi_{\varepsilon \cdot U}(c^j\sigma^k) = (-1)^{j+k}$.

  This gives $1^2 + 1^2 + 1^2 + 1^2 + 2^2\left(\frac{n-2}{2}\right) = 2n$, so again we have all irreps.

Finally, from (*), we actually computed the matrices for these representations, so we know the complete story for the representations of the family D_{2n}.

7) From #3, we know all representations of S_5 either restrict to irreps of A_5 or decompose as $V \oplus \varepsilon \cdot V$.

Noting that the conjugacy class represented by (12345) splits in A_5, this process gives us:

	1 1	15 (12)(34)	20 (123)	12 (12345)	12 (12354)	
χ_1	1	1	1	1	1	← irreducible
χ_2	4	0	1	-1	-1	
χ_3	5	1	-1	0	0	
α	6	-2	0	1	1	← reducible

So $\alpha = \chi_4 + \chi_5$ each 3-dimensional irreps.

Thus we have

	1	15	20	12	12
χ_4	3	a	b	c	d
χ_5	3	$-a-2$	$-b$	$1-c$	$1-d$

since $\chi_4 + \chi_5 = \alpha$

We may use row orthogonality to find $a = -1$, $b = 0$, and c & d satisfy $x^2 = x+1$, so WLOG $c = \frac{1+\sqrt{5}}{2}$, $d = \frac{1-\sqrt{5}}{2}$.

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8) Let α, β be as given, and χ_1 the trivial character.

Can check:
- $\langle \chi_1, \alpha \rangle = 1$, so $\alpha = \gamma + 1$
- $\langle \gamma, \gamma \rangle = 1$, so γ is irreducible
- $\langle \beta, \gamma \rangle = 1$, so $\beta = \delta + \gamma$
- $\langle \delta, \delta \rangle = 1$ with δ a 16-dimensional character.

Thus δ is the desired character, with values:

	1	15	40	90	45	120	144	120	90	15	40
δ	16	0	-2	0	0	0	1	0	0	0	-2

10) Let α, β, γ be as given

i) Can check:
- $\langle \beta, \beta \rangle = 2 = 1^2 + 1^2$, so $\beta = \chi_1 + \chi_2$ distinct irreducible characters
- $\langle \gamma, \gamma \rangle = 4 = 1^2+1^2+1^2+1^2 = 2^2$, so γ contains 4 distinct irreps or one irrep twice
- $2 = \langle \beta, \gamma \rangle = \langle \chi_1, \gamma \rangle + \langle \chi_2, \gamma \rangle$

Suppose $\langle \chi_1, \gamma \rangle = 1$ (so $\langle \chi_2, \gamma \rangle = 1$ as well)

Then γ must contain χ_1 & χ_2, and thus must be a sum of 4 distinct characters. Write $\gamma = \chi_1 + \chi_2 + \xi_1 + \xi_2 = \beta + \xi_1 + \xi_2$

Checking dimension, we see $16 = \dim \gamma = \dim \beta + \dim \xi_1 + \dim \xi_2 \geq 15 + 1 + 1$, which is a contradiction.

Conclude either $\langle \chi_1, \gamma \rangle = 2$ or $\langle \chi_2, \gamma \rangle = 2$. Say it's χ_1. Thus $\gamma = 2\chi_1$,

so we may compute $\chi_1 = \frac{1}{2}\gamma$, irreducible, dimension 8

Thus $\chi_2 = \beta - \chi_1$ is also irreducible, dimension 7

Can check: $\langle \chi_1, \alpha \rangle = 1$, so $\alpha = \chi_1 + \chi_3$

$\langle \chi_3, \chi_3 \rangle = 1$, so χ_3 is irreducible, dimension 6

Including the trivial character, we have 4 irreps, and $1^2 + 6^2 + 7^2 + 8^2 = 150$.

Since there are 6 conjugacy classes, and $|G| = 168$, we know the sum of the squares of the dimensions of the two remaining irreps is 18, so both have dimension 3. Call them χ_4, χ_5.

Using row and column orthogonality relations, we can fill in these rows, resulting in

	1	21	42	56	24	24
Trivial	1	1	1	1	1	1
χ_1	8	0	0	-1	1	1
χ_2	7	-1	-1	1	0	0
χ_3	6	2	0	0	-1	-1
χ_4	3	-1	1	0	$\frac{-1+\sqrt{7}}{2}$	$\frac{-1-\sqrt{7}}{2}$
χ_5	3	-1	1	0	$\frac{-1-\sqrt{7}}{2}$	$\frac{-1+\sqrt{7}}{2}$

ii) Finally, notice that any normal subgroup $H \trianglelefteq G$ leads to irreps of the form $G \to G/H \xrightarrow{\rho} GL_n(\mathbb{C})$, so $\chi(1) = \chi(h)$ for all $h \in H$. The only irrep which has this property for any $g \neq 1$ is the trivial rep, corresponding to $G \trianglelefteq G$. Thus G has no nontrivial normal subgroups.