p 53

20 A quantity and its $\frac{1}{2}$ added together become 16.

Initial guess for false position: 2

$2+1=3$

What multiplies 3 to give 16?

1	3 ✓
2	6
4	12 ✓
$\frac{2}{3}$	2
$\frac{1}{3}$	1 ✓
$5+\frac{1}{3}$	16

The answer is $2(5+\frac{1}{3})$

1	2 ✓
2	4
4	8 ✓
$\frac{2}{3}$	$1+\frac{1}{3}$
$\frac{1}{3}$	$\frac{1}{2}+\frac{1}{6}$ ✓
$5+\frac{1}{3}$	$\boxed{10+\frac{1}{2}+\frac{1}{6}}$

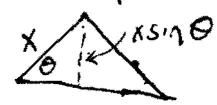
p 62.

9 a) The area of a Δ is given by

$$A=\frac{1}{2}xy\sin\theta \quad (*)$$

where θ is the angle b/w sides with lengths x and y

(for example

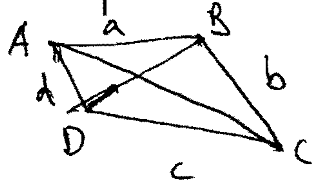


)

Derive the formula

$$A=\frac{1}{4}(ad\sin A+ab\sin B+bc\sin C+cd\sin D) \quad (**)$$

for a quadrilateral



Pf The area is given by

$A=$ area $\Delta ABD+$ area ΔBCD

and

$A=$ area $\Delta ACD+$ area ΔABC.

Adding these equations together and substituting $(*)$ for the areas of the Δs, we get

$$2A=\frac{1}{2}(ad\sin A+ab\sin B+bc\sin C+cd\sin D)$$

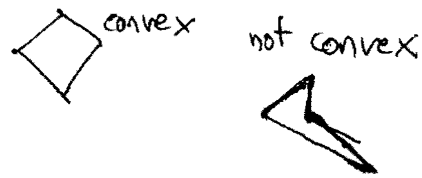
Dividing by 2 gives the formula. □

b) Show that if A represents the area of the quadrilateral in part (a), then $A\le\frac{(a+c)(b+d)}{4}$.

Pf Note that we consider only convex quadrilaterals. This means we must have

$$0<A,B,C,D<\pi$$

(



)

For $0<\theta<\pi$ $0<\sin\theta\le 1$,

so $da\sin A\le da$

$ab\sin B\le ab$ etc.

Thus

$$A=\frac{ab\sin B+ad\sin A+bc\sin C+cd\sin D}{4}$$

$$\le\frac{ad+ab+bc+cd}{4}=\frac{(a+c)(b+d)}{4}$$

□

2 $A=\frac{(a+c)(b+d)}{4}\iff$ the quadrilateral is a rectangle.

pf:

$\Leftarrow$: Suppose the quad. is a rectangle. This means $A=B=C=D=\pi/2$, so $\sin A=\sin B=\sin C=\sin D=1$. Plugging these values into $(**)$, we get

$$A=\frac{ab+ad+bc+cd}{4}$$

$\Rightarrow$: Suppose $A=\frac{(a+c)(b+d)}{4}$

This means

$ab\sin B + ad\sin A + bc\sin C + cd\sin D = ab + bc + cd + ad$.

Further, we know (from above) that $ab\sin B \le ab$ etc.

We must show

$$ab\sin B = ab.$$

Suppose for contradiction that $ab\sin B < ab$.

Then

$ad\sin A + bc\sin C + cd\sin D - (ad+bc+cd) = ab - ab\sin B > 0$.

But this is a contradiction since $\sin\theta \le 1 \Rightarrow ad\sin A + bc\sin C + cd\sin D - (ad+bc+cd) \le 0$.

Thus $ab\sin B \geq ab$.

$\Rightarrow ab\sin B = ab$.

Similarly,

$ad\sin A = ad$
$bc\sin C = bc$
$cd\sin D = cd$.

$\Rightarrow \sin A = \sin B = \sin C = \sin D = 1$

Since we know $0 < A, B, C, D < \pi$, this means $A = B = C = D = \pi/2$, so the quad. is a rectangle. □

(10) a) Of all the Δs with two given sides of lengths a & b, the one whose sides form a right angle encloses the maximum area.

Pf (1) From 9, the area of the Δ is

$A = \frac{1}{2}ab\sin\theta$

where θ is the angle between the sides of lengths a & b.

Since $\sin\theta \le 1$,

$A \le \frac{1}{2}ab$.

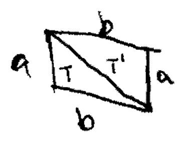
Since $0 < \theta < \pi$ (to form a triangle)

$\sin\theta = 1 \Leftrightarrow \theta = \pi/2$.

Thus the maximum area is ~~achie~~ enclosed by the rt. Δ (Moreover, this is the only angle for which it is obtained.) □

Pf (2)

Given a Δ T of area A, we can obtain a quadrilateral Q by appending a rotated copy of T (call it T') to T



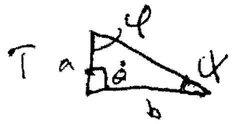
The area of Q is 2A, which is maximised $\Leftrightarrow$ A is maximised.

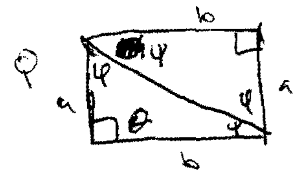
From 9, we know the area of a quadrilateral (with given sides) maximised $\Leftrightarrow$ it is a rectangle.

Claim Q is a rectangle $\Leftrightarrow$ T is a rt. Δ.

($\Rightarrow$: This is clear.

$\Leftarrow$: Suppose T is a rt. Δ. Label the angles:





$\psi + \varphi = \pi - \theta = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

Thus Q is a rectangle.)

The above says, then the area of T is maximised

$\Leftrightarrow$ the area of Q is maximised

$\Leftrightarrow$ Q is a rectangle

$\Leftrightarrow$ T is a right Δ.

□

b) Another formula for the area of a quadrilateral is

$A^2 = (s-a)(s-b)(s-c)(s-d) - T$

where $s = \frac{1}{2}(a+b+c+d)$ and $T = abcd\cos^2(A+C)/2$.

Show this is maximised (given a, b, c, d) when A & C (hence B & D) are supplementary.

Pf Note A is maximised $\Leftrightarrow A^2$ is maximised (since $A>0$)

Also, note that $T \geq 0$. To maximise A^2, we must minimise T, which occurs if $T=0$.

$T=0 \Leftrightarrow \cos^2\left(\frac{A+C}{2}\right)=0 \Leftrightarrow \cos\left(\frac{A+C}{2}\right)=0$

Since we are considering quadrilaterals, this means

$$\frac{A+C}{2} = \pi/2 \Rightarrow A+C=\pi$$

$(\Rightarrow B+D=\pi)$

(If $\frac{A+C}{2} = \frac{3\pi}{2}$, say, then $A+C=3\pi$ ✗ since the sum of the angles of a quadrilateral is 2π)

$\square$

(These proofs are more detailed than I would expect, but remember to write complete sentences and make a coherent argument:

A proofreading " $A=\frac{1}{2}ab\sin\theta$, $-1 \leq \sin\theta \leq 1$, $\sin\frac{\pi}{2}=1$ " for 10_a has necessary elements but is not really satisfactory. ~L)