Section 8.1

2. (a)

\[ f(0) = 0; \]
\[ f'(x) = \frac{1}{1 + x^2}, \quad f'(0) = 1; \]
\[ f''(x) = -\frac{2x}{(1 + x^2)^2}, \quad f''(0) = 0; \]
\[ f^{(3)}(x) = -\frac{2}{(1 + x^2)^2} + \frac{8x^2}{(1 + x^2)^3}, \quad f^{(3)}(0) = -2. \]

Hence

\[ p_3(x) = x - \frac{1}{3} x^3. \]

(c) Since the differentiation operation is linear, and the only derivative of \( x^{200} \) that does not vanish at 0 is its 200th derivative. The first, second, and third derivative of \( f \) at 0 should be the same as that of sin, hence the third Taylor polynomial of \( f \) at 0 is the same as that of sin

\[ p_3(x) = x - \frac{1}{6} x^3. \]

(d)

\[ f(1) = 1; \]
\[ f'(x) = -\frac{1}{2}(2 - x)^{-\frac{1}{2}}, \quad f'(1) = -\frac{1}{2}; \]
\[ f''(x) = -\frac{1}{4}(2 - x)^{-\frac{3}{2}}, \quad f''(1) = -\frac{1}{4}; \]
\[ f^{(3)}(x) = -\frac{3}{8}(2 - x)^{-\frac{5}{2}}, \quad f^{(3)}(1) = -\frac{3}{8}. \]
Hence

\[ p_3(x) = 1 - \frac{1}{2}(x - 1) - \frac{1}{8}(x - 1)^2 - \frac{1}{16}(x - 1)^3. \]

\[ \square \]

4. From the conditions given, we know that \( f(x), f'(x), f''(x) \) are continuous, since all of them are differentiable. Moreover, by the Taylor polynomial, we have

\[ f(0) = 1, \quad f'(0) = 4, \quad f''(0) = -2. \]

Take \( \varepsilon_1 = \frac{1}{2} > 0, \varepsilon_2 = 2 > 0, \varepsilon_3 = 1 > 0. \) By continuity, we have the following

\[ \exists \delta_1 > 0, \text{ such that } \forall x \in (-\delta_1, \delta_1), |f(x) - f(0)| < \varepsilon_1 \]
\[ \Rightarrow f(x) > \frac{1}{2}, \forall x \in (-\delta_1, \delta_1); \]
\[ \exists \delta_2 > 0, \text{ such that } \forall x \in (-\delta_2, \delta_2), |f'(x) - f'(0)| < \varepsilon_2 \]
\[ \Rightarrow f'(x) > 2, \forall x \in (-\delta_2, \delta_2); \]
\[ \exists \delta_3 > 0, \text{ such that } \forall x \in (-\delta_3, \delta_3), |f''(x) - f''(0)| < \varepsilon_3 \]
\[ \Rightarrow f''(x) < -1, \forall x \in (-\delta_3, \delta_3). \]

Let \( \delta = \min\{\delta_1, \delta_2, \delta_3\} > 0, \) then on the interval \( (-\delta, \delta) \) \( f \) has the required property. \[ \square \]

Section 8.2

2. Let \( f(x) = (1 + x)^{\frac{1}{3}} \) by direct computation, we have

\[ f(0) = 1; \]
\[ f'(x) = \frac{1}{3}(1 + x)^{-\frac{2}{3}}, \quad f'(0) = \frac{1}{3}; \]
\[ f''(x) = -\frac{2}{9}(1 + x)^{-\frac{5}{3}}, \quad f''(0) = -\frac{2}{9}; \]
\[ f^{(3)}(x) = \frac{10}{27}(1 + x)^{-\frac{8}{3}}. \]

By Lagrange remainder theorem, we have

\[ f(x) = 1 + \frac{1}{3}x - \frac{1}{9}(1 + c_1)^{-\frac{2}{3}}x^2, \text{ for some } c_1 \in (0, x); \]
\[ f(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{10}{27 \times 3!}(1 + c_2)^{-\frac{8}{3}}x^3, \text{ for some } c_2 \in (0, x). \]

The conclusion follows immediately from the facts that \(-\frac{1}{9}(1 + c_1)^{-\frac{2}{3}}x^2 < 0 \) and \( \frac{10}{27 \times 3!}(1 + c_2)^{-\frac{8}{3}}x^3 > 0. \] \[ \square \]
10. **Only if** By the condition that $f$, $g$ have contact of order $n$, we have $f^{(k)}(0) = g^{(k)}(0)$, for all $k = 0, 1, 2, ..., n$. Writing $f$ and $g$ as $n$th Taylor polynomial with Lagrange remainder, we have

$$f(x) - g(x) = \frac{1}{(n + 1)!} \left( f^{(n+1)}(c_1) - g^{(n+1)}(c_2) \right) x^{n+1},$$
for some $c_1, c_2 \in (0, x)$ or $(x, 0)$.

It follows immediately that

$$\lim_{x \to 0} f(x) - g(x) = \lim_{x \to 0} \frac{1}{(n + 1)!} \left( f^{(n+1)}(c_1) - g^{(n+1)}(c_2) \right) x = \frac{1}{(n + 1)!} \left( f^{(n+1)}(0) - g^{(n+1)}(0) \right) \times 0 = 0.
$$

If An argument by contradiction. Suppose the statement is not true, let $k$ be the smallest natural number such that $f^{(k)}(0) \neq g^{(k)}(0)$. By the contradiction assumption, $k \leq n$. Writing $f$, $g$ as $k - 1$th Taylor polynomial with Lagrange remainder centered at 0, we have

$$f(x) - g(x) = \frac{1}{k!} (f^{(k)}(c_1) - g^{(k)}(c_2)) x^k,$$
for some $c_1, c_2 \in (0, x)$ or $(x, 0)$. Notice here we have used the assumption that $f^{(i)}(0) = g^{(i)}(0)$ for all $i \leq k - 1$. It follows that

$$\lim_{x \to 0} \frac{f(x) - g(x)}{x^n} = \lim_{x \to 0} \frac{1}{k!} (f^{(k)}(c_1) - g^{(k)}(c_2)) x^{k-n}$$

where $f^{(k)}(c_1) - g^{(k)}(c_2)$ are bounded away from 0 when $c_1, c_2$ are in a small neighbourhood of 0, because $f^{(k)}(0) \neq g^{(k)}(0)$ and because of the continuity. Moreover either $x^{k-n}$ converges to 1 (in the case $k = n$) or $|x|^{k-n}$ diverges to $\infty$ (in the case $k < n$). Therefore the limit $\lim_{x \to 0} \frac{f(x) - g(x)}{x^n}$ either does not exist or is not 0, a contradiction.

Section 8.3

3. Differentiating the first given inequality for $n$ times we have $F^{(n+1)}(x) = F^{(n)}(x)$, for all $x$. Hence $F^{(n)}(0) = F(0) = 2$, for all $n$. Thus the $n$th Taylor polynomial is

$$p_n(x) = \sum_{k=0}^{n} \frac{2}{k!} x^k.$$

We apply theorem 8.14 to show that the Taylor expansion converges to $F$ for any $x \in \mathbb{R}$. Fix an arbitrary $x \in \mathbb{R}$, let $r > 0$ be such that $x \in [-r, r]$, we need only to show the Taylor
expansion converges on the interval \([-r, r]\), since \(x\) is arbitrarily chosen. \(F\) is continuous on \([-r, r]\), hence it is bounded on this interval. Let \(M > 0\) be such that \(|F(x)| \leq M\), for all \(x \in [-r, r]\). Without loss of generality, we assume \(M \geq 1\). By the first equality in the problem we know that \(F^{(n+1)}(x) = F^{(n)}(x) = F^{(n-1)}(x) = ... = F(x)\), for all \(n \in \mathbb{N}\) and for all \(x \in \mathbb{R}\). Therefore

\[ |F^{(n)}(x)| = |F(x)| \leq M \leq M^n, \text{ for all } n \geq 1, \; x \in [-r, r]. \]

By theorem 8.14, the Taylor expansion converges on \([-r, r]\).

Section 8.5

5. The Cauchy remainder is

\[ \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt. \]

By the assumption that \(f^{(n+1)}\) is continuous. Moreover, when \(t \in [x, x_0]\) or \(t \in [x_0, x]\), \((x-t)^n\) is either nonpositive or nonnegative. For the nonnegative case we will apply exercise 8.5.4 and for the non positive case we have

\[ \int_{a}^{b} h(x)g(x)dx = - \left( \int_{a}^{b} (-h(x))g(x)dx \right) = g(c) \int_{a}^{b} h(x)dx, \]

where the second equality is due to exercise 8.5.4. Therefore

\[ \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt = \frac{1}{n!} f^{(n+1)}(c) \int_{x_0}^{x} (x-t)^n dt, \text{ for some } c \in (x_0, x) \]

\[ = \frac{1}{n!} f^{(n+1)}(c) \frac{1}{n+1} (x-t)^{n+1}|_{t=x_0}^{x} \]

\[ = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-x_0)^{n+1}, \text{ for some } c \in (x_0, x). \]

Which is the Lagrange remainder.

Appendix. Proof of exercise 8.5.4

Proof. Let \(I = \int_{a}^{b} h(x)dx\), we may assume that \(I > 0\), because otherwise since \(h \geq 0\) and by a standard argument of continuity we may obtain that \(h \equiv 0\), and the statement becomes trivial. Now the statement becomes that there exists \(c \in (a, b)\) such that

\[ g(c) = \frac{1}{I} \int_{a}^{b} h(x)g(x)dx.\]
Let \( x_m, x_M \in [a, b] \) be such that \( m = g(x_m) = \min_{x \in [a, b]} g(x), M = g(x_M) = \max_{x \in [a, b]} g(x) \).
We may also assume \( m < M \) because otherwise \( g \) is constant and the statement is trivial again. We have
\[
m = \frac{1}{I} \int_a^b mh(x)dx \leq \frac{1}{I} \int_a^b g(x)h(x)dx \leq \frac{1}{I} \int_a^b Mh(x)dx = M,
\]
(1)
since \( h \geq 0 \). Therefore, \( \exists c \in [a, b] \) such that \( g(c) = \frac{1}{I} \int_a^b h(x)g(x)dx \). Now we want to prove that we can take \( c \) in the open interval \((a, b)\). Suppose this is not true, that is \( g(c) \neq \frac{1}{I} \int_a^b h(x)g(x)dx \) for all \( c \in (a, b) \), then by intermediate value theorem, either \( g(c) > \frac{1}{I} \int_a^b h(x)g(x)dx \) for all \( c \in (a, b) \), or \( g(c) < \frac{1}{I} \int_a^b h(x)g(x)dx \) for all \( c \in (a, b) \). We assume without loss of generality that the first one is the case. Then
\[
g(x) - \frac{1}{I} \int_a^b h(t)g(t)dx > 0, \text{ for all } x \in (a, b)
\]
Furthermore, since \( h \geq 0 \), not identically 0 and continuous, we know that \( \exists x_0 \in (a, b) \), such that \( h(x_0) > 0 \), hence
\[
h(x) \left( g(x) - \frac{1}{I} \int_a^b h(t)g(t)dx \right) \geq 0, \text{ for all } x \in (a, b)
\]
\[
h(x_0) \left( g(x_0) - \frac{1}{I} \int_a^b h(t)g(t)dx \right) > 0.
\]
By a standard argument of continuity, we have
\[
\int_a^b h(x) \left( g(x) - \frac{1}{I} \int_a^b h(t)g(t)dx \right) dx > 0,
\]
that is,
\[
\int_a^b h(x)g(x)dx - \int_a^b h(t)g(t)dx > 0,
\]
which is a contradiction.

Moreover, if \( h \leq 0 \) on \([a, b]\), we have \( \exists c \in (a, b) \), such that
\[
\int_a^b h(x)g(x)dx = - \left( \int_a^b (-h(x))g(x)dx \right) = - \left( g(c) \int_a^b (-h(x))dx \right) = g(c) \int_a^b h(x)dx.
\]
\[\square\]