2. Observe that
\[
\lim_{k \to \infty} \frac{(k + 1)\alpha e^{-(k+1)}}{k^\alpha e^{-k}} = \lim_{k \to \infty} \left(1 + \frac{1}{k}\right) \alpha \cdot \frac{1}{e} = \frac{1}{e} < 1,
\]
it follows from the **Ratio Test** that the series converges.

8. By definition of limit, we know that there exists \(N \in \mathbb{N}\) large, such that for all \(k \geq N\)
\[
\frac{a_k}{b_k} \leq 2l \quad \Rightarrow \quad 0 \leq a_k \leq 2lb_k.
\]

Moreover, we observe that \(\sum_{k=1}^{n} 2lb_k = 2l \sum_{k=1}^{n} b_k\), hence the convergence of positive series \(\sum_{k=1}^{\infty} b_k\) implies the convergence of the positive series \(\sum_{k=1}^{\infty} 2lb_k\). It follows from Corollary 9.8 that \(\sum_{k=1}^{\infty} a_k\) converges. The converse follows in the same way from that
\[
\lim_{k \to \infty} \frac{b_k}{a_k} = \frac{1}{l}.
\]
3. Fix any \( x \in (0, 1) \), one may easily observe that

\[
\lim_{n \to \infty} f_n(x) = \lim_{x \to \infty} \frac{1}{nx + 1} = \lim_{x \to \infty} \frac{1}{x + \frac{1}{n}} = \frac{0}{x} = 0.
\]

Hence \( f(x) = 0 \), \( x \in (0, 1) \) is the pointwise limit of the sequence \( \{f_n(x), x \in (0, 1)\}_{n=1}^{\infty} \). Moreover, it is easy to check that \( f_n(\frac{1}{n}) = \frac{1}{2} \), hence

\[
\sup_{x \in (0,1)} |f_n(x) - f(x)| \geq \frac{1}{2^r},
\]

and the convergence is not uniform.

\[\square\]

4. Revised version: \( f_n(x) = \frac{x}{nx^2 + 1} \).

Obviously \( f_n(0) = 0 \) for all \( n \in \mathbb{N} \). Fix any \( x \in (0, 1] \), we have

\[
\lim_{n \to \infty} \frac{x}{nx^2 + 1} = \lim_{n \to \infty} \frac{x}{n + \frac{1}{x^2}} = \frac{0}{x^2} = 0.
\]

Therefore \( f(x) = 0 \), \( x \in [0, 1] \) is the pointwise limit of the sequence \( \{f_n(x), x \in [0, 1]\}_{n=1}^{\infty} \). Now we show that the convergence is uniform in \( x \). Fix any \( \varepsilon > 0 \), we have

\[
\frac{x}{nx^2 + 1} < \varepsilon, \quad \text{whenever } x < \varepsilon \text{ and for all } n \in \mathbb{N}.
\]  \hspace{1cm} (1)

Moreover

\[
\frac{x}{nx^2 + 1} \leq \frac{1}{n\varepsilon^2 + 1} < \varepsilon, \quad \text{whenever } x \in [\varepsilon, 1] \text{ and for all } n > \frac{1-\varepsilon}{\varepsilon^3}.
\]  \hspace{1cm} (2)

Therefore, fixing a \( N \in \mathbb{N} \) such that \( N > \frac{1-\varepsilon}{\varepsilon^3} \), combining (1) and (2) we have

\[
0 \leq \frac{x}{nx^2 + 1} < \varepsilon, \quad \text{whenever } n > N \text{ and for all } x \in [0, 1],
\]

and hence uniform convergence.

\[\square\]

7. Let \( \varepsilon = 1 \), by definition of uniform convergence, there exists \( n \), such that \( |f_n(x) - f(x)| \leq 1 \), for all \( x \in \mathbb{R} \). Therefore \( |f(x)| \leq |f_n(x)| + 1 \) for all \( x \in \mathbb{R} \) and \( f(x) \) is bounded since \( f_n(x) \) is.

\[\square\]