Subword Languages of Infinite Partial Words

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1. Preliminaries
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Formal Languages

Let $L \subset A^*$ be a language over the finite alphabet $A$. 
Formal Languages

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Let $L \subseteq A^*$ be a language over the finite alphabet $A$. We say $L$ is factorial if, whenever $u \in L$, then $v \in L$ for every subword $v$ of $u$. We say that a word $u \in L$ is left-prolongable (resp. right-prolongable) if there exists $a \in A$ so that $au \in L$ (resp. $ua \in L$). If $u$ is left- and right-prolongable then we say $u$ is prolongable.
Formal Languages

Let $L \subseteq A^*$ be a language over the finite alphabet $A$. We say $L$ is factorial if, whenever $u \in L$, then $v \in L$ for every subword $v$ of $u$. We say that a word $u \in L$ is left-prolongable (resp. right-prolongable) if there exists $a \in A$ so that $au \in L$ (resp. $ua \in L$). If $u$ is left- and right-prolongable then we say $u$ is prolongable. We say $L$ is left-prolongable (resp. right-prolongable, prolongable) if every $u \in L$ is.
Formal Languages

Example

Let $A = \{a, b\}$. 
**Example**

Let $A = \{a, b\}$.

$L = \{a, ab, abb, abba, abbaa, abbaaa, abaaaa, \ldots \}
\quad b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots
\quad aa, aaa, aaaa, aaaaa, aaaaaa, \ldots
\quad bb, bba, bbba, bbbaa, bbbaaa, bbbaaaa, \ldots
\quad aba, abaa, abaaa, abaaaa, \ldots
\quad baa, baaa, baaaa, baaaaa, \ldots\}$
Formal Languages

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Let $A = \{a, b\}$.

$L = \{a, ab, abb, abba, abbaa, abbaaa, abbaaaa, \ldots\}
\quad \quad \quad \quad \quad b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots$
\quad \quad \quad \quad \quad aa, aaa, aaaa, aaaaa, aaaaaa, \ldots
\quad \quad \quad \quad \quad bb, bba, bbaa, bbaba, bbaaa, bbaaaa, \ldots
\quad \quad \quad \quad \quad aba, abaa, abaaa, abaaaa, \ldots
\quad \quad \quad \quad \quad baa, baaa, baaaa, baaaaa, \ldots\}$
Formal Languages

Example

Let $A = \{a, b\}$.

\[
L = \{ a, ab, abb, abba, abbaa, abbaaa, abaaaa, \ldots \\
b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \\
aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \\
bb, bba, bbba, bbaaa, bbaaaa, \ldots \\
aba, abaa, abaaa, abaaaa, \ldots \\
baa, baaa, baaaa, baaaaa, \ldots \}
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b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots$

aa, aaa, aaaa, aaaaa, aaaaaa, \ldots$

bb, bba, bbaa, bbaaa, bbaaaa, \ldots$

aba, abaa, abaaa, abaaaa, \ldots$

baa, baaa, baaaa, baaaaa, \ldots\}$
Formal Languages

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\quad \{b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \}
\quad \{aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \}
\quad \{bb, bba, bbba, bbbaa, bbbaaa, bbbaaaa, \ldots \}
\quad \{aba, abaa, abaaa, abaaaa, \ldots \}
\quad \{baa, baaa, baaaa, baaaaa, \ldots \}\}$$
Formal Languages

Example

Let $A = \{a, b\}$.

\[ L = \{a, ab, abb, abba, abbaa, abbaaa, abaaaa, \ldots \}
\[ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \]

\[ aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \]

\[ bb, bba, bbaa, bbaaa, bbaaaa, \ldots \]

\[ aba, abaa, abaaa, abaaaa, \ldots \]

\[ baa, baaa, baaaa, baaaaa, \ldots \} \]
Formal Languages

Example

Let $A = \{a, b\}$.

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$bb, bba, bbba, bbbaa, bbbaaa, \ldots$

$aba, abaa, abaaa, abaaaa, \ldots$

$baa, baaa, baaaa, baaaaa, \ldots$\}$
Formal Languages

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$b, ba, bab, babaa, babaaa, babaaaa, \ldots$
$aa, aaa, aaaa, aaaaa, aaaaaa, \ldots$
$bb, bba, bbaa, bbaaa, bbaaa, \ldots$
$aba, abaa, abaaa, abaaaa, \ldots$
$baa, baaa, baaaa, baaaaa, \ldots\}$
Formal Languages

Example

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$$L = \{ a, ab, abb, abba, abbaa, abbaaa, abaaaa, \ldots \\
    b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \\
    aa, aaa,aaaa, aaaaaa, aaaaaaa, \ldots \\
    bb, bba, bbba, bbbaa, bbbaaa, bbbaaaa, \ldots \\
    aba, abaa, abaaa, abaaaaa, \ldots \\
    baa, baaa, baaaa, baaaaa, \ldots \}$$
Formal Languages

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$L = \{a, ab, abb, abba, abbaa, abbaaa, abbaaaa, \ldots$

$b, ba, bab, baba, babaa, babaaa, \ldots$

$aa, aaa, aaaa, aaaaa, aaaaaa, \ldots$

$bb, bba, bbaa, bbaaa, bbaaaa, \ldots$

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Formal Languages

Example

Let $A = \{a, b\}$.

$$L = \{a, ab, abb, abba, abbaa, abbaaa, abbaaaa, \ldots \}$$

$$b, ba, bab, \textcolor{red}{baba}, \textcolor{red}{babaa}, babaaa, babaaaa, \ldots$$

$$aa, aaa, aaaa, aaaaa, aaaaaa, \ldots$$

$$bb, bba, bbaa, bbaaa, bbaaaa, \ldots$$

$$aba, abaa, abaaa, abaaaa, \ldots$$

$$baa, baaa, baaaa, baaaaa, \ldots \}$$
Partial Words

Let \( w \) be a partial word.
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The greatest lower bound of \( u, v \) is the word with \( \diamond \) whenever \( u, v \) disagree or are \( \diamond \).
Partial Words

Let $w$ be a partial word.
A completion of $w$ is a compatible full word.
The greatest lower bound of $u, v$ is the word with $\Diamond$ whenever $u, v$ disagree or are $\Diamond$.

Example

$u = abaa\Diamond \quad v = bba\Diamond b$
Partial Words

Let $w$ be a partial word.

A **completion** of $w$ is a compatible full word.

The **greatest lower bound** of $u, v$ is the word with $\Diamond$ whenever $u, v$ disagree or are $\Diamond$.

**Example**

$$u = abaa\Diamond \quad v = bba\Diamond b$$

$$u \land v = \Diamond ba\Diamond$$
Subword Languages

Let \( w \) be a right infinite partial word.
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Subword Languages

Let $w$ be a right infinite partial word. We call $\text{Sub}(w)$ the subword language of $w$. If a language $L = \text{Sub}(w)$ for some $w$, we say $L$ is representable and $w$ is a representing word for $L$. 
Subword Languages

Let \( w \) be a right infinite partial word. We call \( \text{Sub}(w) \) the subword language of \( w \). If a language \( L = \text{Sub}(w) \) for some \( w \), we say \( L \) is representable and \( w \) is a representing word for \( L \).

Example

Let \( w = \diamond aaaa \cdots \). Then

\[
\text{Sub}(w) = \{ b, a, ba, aa, baa, aaa, \ldots \}
\]
Subword Languages

Any subword language of a set of words is factorial and right-prolongable.
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Any subword language of a set of words is factorial and right-prolongable.

Proposition

Let $L$ be factorial and right-prolongable. Then $L = \text{Sub}(S)$ for some set $S$ of infinite words.
Subword Languages

Any subword language of a set of words is factorial and right-prolongable.

**Proposition**

*Let* \( L \) *be factorial and right-prolongable.*

*Then* \( L = \text{Sub}(S) \) *for some set* \( S \) *of infinite words.*

**Proof.**

Throw a word into \( S \), right-prolong it countably many times, and repeat.
Subword Languages

Any subword language of a set of words is factorial and right-prolongable.

**Proposition**

Let $L$ be factorial and right-prolongable. Then $L = \text{Sub}(S)$ for some set $S$ of infinite words.

**Proof.**

Throw a word into $S$, right-prolong it countably many times, and repeat.

**Question**

When does there exist a $w$ so that $L = \text{Sub}(w)$?
Recurrent Words

An infinite partial word $w$ is recurrent if every subword appears infinitely many times.
Recall that a partial word \( w \) is recurrent if every subword appears infinitely many times.

If \( \sigma_n(w) = w_n w_{n+1} w_{n+2} \cdots \) is recurrent for some \( n \), then we say \( w \) is ultimately recurrent.
An infinite partial word $w$ is **recurrent** if every subword appears infinitely many times. If $\sigma_n(w) = w_n w_{n+1} w_{n+2} \cdots$ is recurrent for some $n$, then we say $w$ is **ultimately recurrent**.

**Proposition**

Let $w$ be an infinite partial word. The following are equivalent.
Recurrent Words

An infinite partial word \( w \) is recurrent if every subword appears infinitely many times.

If \( \sigma_n(w) = w_n w_{n+1} w_{n+2} \cdots \) is recurrent for some \( n \), then we say \( w \) is ultimately recurrent.

**Proposition**

Let \( w \) be an infinite partial word. The following are equivalent.

(a) \( w \) is recurrent.

(b) \( \text{Sub}(\sigma_n(w)) = \text{Sub}(w) \) for every \( n \).

(c) Every subword of \( w \) occurs at least twice.

(d) Every prefix of \( w \) occurs at least twice.
Decreasing Sequences

**Definition**

A **decreasing sequence** is a sequence $u_i$ of partial words such that the sequence $|u_i|$ is monotonically increasing, and $u_n \supset u_m[0,|u_n| - 1)$ whenever $m \geq n$. 

Lemma (Monotone Convergence.)

Let $u_n$ be a decreasing sequence of partial words. Then $u_n$ converges (pointwise) to an infinite partial word $w$, and every subword of a $u_n$ is a subword of $w$. 

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Partial Subword Languages
**Decreasing Sequences**

**Definition**

A *decreasing sequence* is a sequence $u_i$ of partial words such that the sequence $|u_i|$ is monotonically increasing, and $u_n \supset u_m[0,|u_n| - 1)$ whenever $m \geq n$.

**Lemma (Monotone Convergence.)**

Let $u_n$ be a decreasing sequence of partial words. Then $u_n$ converges (pointwise) to an infinite partial word $w$, and every subword of a $u_n$ is a subword of $w$. 
Decreasing Sequences

Example

Consider the sequence

\[ a, ab, aba, \cdots ba, \cdots ba, \cdots a, \cdots ba, \cdots \]

The pointwise limit is the word \( w = \cdots ba, \cdots ba, \cdots a, \cdots ba, \cdots \).
Decreasing Sequences

Example

Consider the sequence

\[ a, \]

The pointwise limit is the word \( w \).
Decreasing Sequences

Example

Consider the sequence

\[ a, ab, \]
Decreasing Sequences

Example

Consider the sequence

\[ a, ab, aba, \]
Decreasing Sequences

Example

Consider the sequence

\[ a, ab, aba, \diamond baa, \]
Decreasing Sequences

Example

Consider the sequence

\[ a, ab, aba, \diamond baa, \diamond ba\diamond b, \]
Decreasing Sequences

Example

Consider the sequence

$$a, ab, aba, \Diamond baa, \Diamond ba\Diamond b, \Diamond\Diamond a\Diamond ba, \ldots$$
Decreasing Sequences

Example

Consider the sequence

\[ a, ab, aba, \Diamond baa, \Diamond ba\Diamond b, \Diamond \Diamond a\Diamond ba, \ldots \]

The pointwise limit is the word

\[ w = \]
Decreasing Sequences

Example

Consider the sequence

\[ a, ab, aba, \Diamond baa, \Diamond ba\Diamond b, \Diamond\Diamond a\Diamond ba, \ldots \]

The pointwise limit is the word

\[ w = a \]
Decreasing Sequences

Example

Consider the sequence

\[ a, ab, aba, \Diamond baa, \Diamond ba\Diamond b, \Diamond \Diamond a\Diamond ba, \ldots \]

The pointwise limit is the word

\[ w = ab \]
Decreasing Sequences

Example

Consider the sequence

\[ a, ab, aba, \diamond baa, \diamond ba\diamond b, \diamond \diamond a\diamond ba, \ldots \]

The pointwise limit is the word

\[ w = aba \]
Decreasing Sequences

Example

Consider the sequence

\[ a, ab, aba, \Diamond baa, \Diamond ba\Diamond b, \Diamond \Diamond a\Diamond ba, \ldots \]

The pointwise limit is the word

\[ w = \Diamond baa \]
Decreasing Sequences

Example

Consider the sequence

\[ a, ab, aba, \Diamond baa, \Diamond ba\Diamond b, \Diamond \Diamond a\Diamond ba, \ldots \]

The pointwise limit is the word

\[ w = \Diamond ba\Diamond b \]
Decreasing Sequences

Example

Consider the sequence

\[ a, ab, aba, baa, ba\diamond b, a\diamond ba, \ldots \]

The pointwise limit is the word

\[ w = a\diamond ba \]
Decreasing Sequences

**Example**

Consider the sequence

\[ a, ab, aba, \diamond baa, \diamond ba\diamond b, \diamond\diamond a\diamond ba, \ldots \]

The pointwise limit is the word

\[ w = \diamond\diamond a\diamond ba \cdots \]
NLP Subwords

Lemma

A subword language $\text{Sub}(w)$ contains nonleft-prolongable words (hereafter NLP) if and only if $w$ is nonrecurrent.
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A subword language \( \text{Sub}(w) \) contains nonleft-prolongable words (hereafter NLP) if and only if \( w \) is nonrecurrent.

Lemma

Let \( w \) be an infinite word and let \( u \in \text{Sub}(w) \) be NLP. Then \( u \) is a prefix of \( w \) and occurs nowhere else.
NLP Subwords

Lemma

A subword language Sub(w) contains nonleft-prolongable words (hereafter NLP) if and only if w is nonrecurrent.

Lemma

Let w be an infinite word and let u ∈ Sub(w) be NLP. Then u is a prefix of w and occurs nowhere else.

Lemma

Let L be a factorial language and let u be NLP. For all x ∈ A*, if ux ∈ L, then ux is NLP.
Nonrecurrent Full Words

Theorem

Let $L$ be a language. Then $L = \text{Sub}(w)$ for a nonrecurrent full word $w$ if and only if

(a) $L$ is factorial and right-prolongable;

(b) $L$ contains at most one NLP subword of each length; and

(c) for each $s \in L$, there exists $x \in A^*$ (possibly empty) so that $xs \in L$ and is NLP.
The Obvious Necessary Conditions...

Proof of (a).

\[ w = abbabbbabaaba \cdots \]
The Obvious Necessary Conditions...

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\[ w = abbababbbabaaba \cdots \]
The Obvious Necessary Conditions...

Proof of (a).

\[ w = abbabbbabaaba \ldots \]
The Obvious Necessary Conditions...

Proof of (a).

\[ w = \textit{abbababbbabaaba} \cdots \]
The Obvious Necessary Conditions...

Proof of (a).

\[ w = abbababbbabaaba \cdots \]
The Obvious Necessary Conditions...

Proof of (a).

\[ w = abbabbbabaaba \cdots \]

Proof of (c).

The subword \( u \) appears finitely many times.

\[ w = w_0 w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_9 w_{10} w_{11} \cdots \]
The Obvious Necessary Conditions...

Proof of (a).

\[ w = abbababbbabaaba \cdots \]

Proof of (c).

The subword \( u \) appears finitely many times.

\[ w = w_0 w_1 \overbrace{w_2 w_3 w_4 w_5 w_6 w_7}^{u} w_8 w_9 w_{10} w_{11} \cdots \]
The Obvious Necessary Conditions...

Proof of (a).

\[ w = abbababbbabaaba \cdots \]

Proof of (c).

The subword \( u \) appears finitely many times.

\[ w = w_0 w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_9 w_{10} w_{11} \cdots \]
The Obvious Necessary Conditions...

Proof of (a).

\[ w = abbababbabaaba \cdots \]

Proof of (c).

The subword \( u \) appears infinitely many times.

\[ W = w_0 w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_9 w_{10} w_{11} \cdots \]
The Obvious Necessary Conditions...

Proof of (a).

\[ w = abbababbbabaaba \ldots \]

Proof of (c).

The subword \( u \) appears infinitely many times.

\[ w = \underbrace{w_0 w_1 w_2 w_3 w_4 w_5 w_6}_\text{NLP} w_7 w_8 w_9 w_{10} w_{11} \ldots \]
The Obvious Necessary Conditions...

Proof of (a).

\[ w = abbabbbbaaba \cdots \]

Proof of (c).

The subword \( u \) appears infinitely many times.

\[ w = w_0 w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_9 w_{10} w_{11} \cdots \]

\( u \)
The Obvious Necessary Conditions...

Proof of (a).

\[ w = abbababbabaaba \cdots \]

Proof of (c).

The subword \( u \) appears infinitely many times.

\[ w = w_0 w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_9 w_{10} w_{11} \cdots \]

\[ u \]
\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \] 
\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \] 
\[ aa, aaa,aaaa, aaaaaa, aaaaaaa, \ldots \] 
\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]
...Are Also Sufficient!

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \] 
\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \] 
\[ aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \] 
\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]
Nonrecurrent Words

...Are Also Sufficient!

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \} \]
\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \]
\[ aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \]
\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]

\[ w = b \]
...Are Also Sufficient!

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \} \]
\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \]
\[ aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \]
\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]

\[ w = ba \]
... Are Also Sufficient!

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \]  
\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \]  
\[ aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \]  
\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]

\[ w = bab \]
Nonrecurrent Words

... Are Also Sufficient!

\[ L = \{ b, ba, bab, baba, babaa, babaaa, \ldots \\
     a, ab, aba, abaa, abaaa, abaaaa, \ldots \\
     aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \\
     baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]

\[ w = baba \]
Nonrecurrent Words

... Are Also Sufficient!

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \]  
\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \]  
\[ aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \]  
\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]

\[ w = babaa \]
\[ L = \{ b, ba, bab, baba, babaa, babaaa, baaaaa, \ldots \} \]
\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \]
\[ aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \]
\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]
\[ w = babaaa \]
Nonrecurrent Words

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$L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \}
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\{ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \}\

w = babaaaa
...Are Also Sufficient!

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\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \] 
\[ aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \] 
\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]

\[ w = babaaaa \ldots \]
Nonrecurrent Words

... Are Also Sufficient!

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \} \]
\[ \quad a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \]
\[ \quad aa, aaa, aaaa, aaaaaa, aaaaaaaaa, \ldots \]
\[ \quad baa, baaa, baaaa, baaaaa, baaaaaaaa, \ldots \} \]

\[ w = babaaaa \ldots \]
Nonrecurrent Words

...Are Also Sufficient!

\[ L = \{ b, ba, bab, baba, babaa, \textit{babaaa}, babaaaa, \ldots \}
\]
\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \]
\[ aa, aaa, aaaa, aaaaaa, aaaaaaa, \ldots \]
\[ baa, baaa, baaaa, baaaaa, baaaaaaaa, \ldots \} \]

\[ w = \textit{babaaaa} \ldots \]
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\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \]
\[ aa, aaa, aaaa, aaaaaa, aaaaaaa, \ldots \]
\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]

\[ w = babaaa \ldots \]
\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \} \]
\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \]
\[ aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \]
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\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]

\[ w = babaaaa \ldots \]
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\[ aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \]

\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]

\[ w = babaaaa \cdots \]
Nonrecurrent Words

... Are Also Sufficient!

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \]  
\[ a, ab, aba, abaa, abaaa, abaaaa, babaaaaa, \ldots \]  
\[ aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \]  
\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \}\]

\[ w = babaaaa \ldots \]
Nonrecurrent Words

... Are Also Sufficient!

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \}
    a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots 
    aa, aaa, aaaa, aaaaa, aaaaaa, \ldots 
    baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]

\[ w = babaaaa \ldots \]
Theorem

Let $L$ be a language, and let $\Gamma_n$ be the set of all NLP words of length $n$. Let $\gamma_n$ be the greatest lower bound of $\Gamma_n$. Then $L = \text{Sub}(w)$ for a nonrecurrent partial word $w$ if and only if

(a) $L$ is factorial and right-prolongable;

(b) every completion of $\gamma_n$ belongs to $L$; and

(c) for each $s \in L$, there exists $x \in A^*$ (possibly empty) so that $xs \in \Gamma_n$. 

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UNCG, UCSD, Princeton
Nonrecurrent Partial Words

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \}
\]
\[ \ldots aab, aaba, aabaa, aabaaa, aabaaaa, \ldots \]
\[ \ldots a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \]
\[ \ldots aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \]
\[ \ldots baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]
Nonrecurrent Partial Words

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \]
\[ aab, aaba, aabaa, aabaaa, aabaaaa, \ldots \]
\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \]
\[ aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \]
\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]
Nonrecurrent Partial Words

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \\
    aab, aaba, aabaa, aabaaa, aabaaaa, \ldots \\
    a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \\
    aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \\
    baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]

\[ w = b \]
Nonrecurrent Partial Words

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \}

\[ aab, aaba, aabaa, aabaaa, aabaaaa, \ldots \]

\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \]

\[ aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \]

\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]

\[ w = ba \]
Nonrecurrent Partial Words

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \]  
\[ aab, aaba, aabaa, aabaaa, aabaaaa, \ldots \]  
\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \]  
\[ aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \]  
\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]

\[ w = \diamond ab \]
Nonrecurrent Partial Words

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \] 
\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
Nonrecurrent Partial Words

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \]
\[ aab, aaba, aabaa,aabaaa, aabaaaa, \ldots \]
\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \]
\[ aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \]
\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]

\[ w = \Diamond abaa \]
Nonrecurrent Partial Words

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \]  
\[ aab, aaba, aabaa, aabaaa, aabaaaa, \ldots \]  
\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \]  
\[ aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \]  
\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]

\[ w = \Diamond abaaa \]
Nonrecurrent Partial Words

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \]  
\[ aab, aaba, aabaa, aabaa, aabaaa, aabaaaa, \ldots \]  
\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \]  
\[ aa, aaa, aaaa, aaaaaa, aaaaaaa, \ldots \]  
\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \} \]

\[ w = \Diamond abaaaa \]
Nonrecurrent Partial Words

\[ L = \{ b, ba, bab, baba, babaa, babaaa, babaaaa, \ldots \}
\]
\[ aab, aaba, aabaa, aabaaa, aabaaaa, \ldots \]
\[ a, ab, aba, abaa, abaaa, abaaaa, abaaaaa, \ldots \]
\[ aa, aaa, aaaa, aaaaa, aaaaaa, \ldots \]
\[ baa, baaa, baaaa, baaaaa, baaaaaa, \ldots \}\]

\[ w = \Diamond abaaaa \ldots \]
**Theorem**

Let $L$ be a language. Then $L = \text{Sub}(w)$ for a recurrent full word $w$ if and only if

(a) $L$ is factorial; and

(b) for $x, z \in L$, there exists $y \in A^*$ so that $xyz \in L$. 
The Obvious Necessary Conditions...

\[ W = w_0 w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_9 w_{10} w_{11} \cdots \]
The Obvious Necessary Conditions...

\[ w = w_0 \underbrace{w_1 w_2 w_3}_x w_4 w_5 w_6 w_7 w_8 w_9 w_{10} w_{11} \cdots \]
The Obvious Necessary Conditions...

\[ w = w_0 w_1 w_2 w_3 \underbrace{w_4 w_5 w_6 w_7}_{x} w_8 w_9 \underbrace{w_{10} w_{11} \cdots}_{z} \]
The Obvious Necessary Conditions...

\[ w = w_0 w_1 w_2 w_3 \quad w_4 w_5 w_6 w_7 \quad w_8 w_9 w_{10} w_{11} \cdots \]

\[ x \quad y \quad z \]
...Are Also Sufficient!

$L = \{a, ab, bba, ba, \ldots \}
\quad abab, ababaabba, ababaabbbaba, \ldots \}$
...Are Also Sufficient!

\[ L = \{ a, ab, bba, ba, \ldots \} \]
\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad abab, ababaabba, ababaabbaba, \ldots \} \]
... Are Also Sufficient!

\[ L = \{ a, ab, bba, ba, \ldots \} \]

\[ abab, ababaabba, ababaabbaba, \ldots \} \]
...Are Also Sufficient!

\[ L = \{ a, ab, bba, ba, \ldots \} \]

\[ \ldots abab, ababaabba, ababaabbbaba, \ldots \} \]
\[ L = \{ a, ab, bba, ba, \ldots \} \]

\[ abab, ababaabba, ababaabbbaba, \ldots \]
...Are Also Sufficient!

\[ L = \{ a, ab, bba, ba, \ldots \} \]

\[ \begin{align*}
abab, & \quad ababaabba, \quad ababaabbbaba, \ldots \\
\end{align*} \]

\[ w = \]
... Are Also Sufficient!

$L = \{ a, ab, bba, ba, \ldots \}

abab, ababaabba, ababaabbbaba, \ldots \}$

$w =$
...Are Also Sufficient!

\[ L = \{ a, ab, bba, ba, \ldots \} \]

\[ w = a \]
...Are Also Sufficient!

\[ L = \{a, ab, bba, ba, \ldots\} \]

\[ abab, ababaabba, ababaabbaba, \ldots \}\]

\[ w = a \]
... Are Also Sufficient!

\[ L = \{ a, ab, bba, ba, \ldots \} \]

\[ abab, ababaabba, ababaabbaba, \ldots \} \]

\[ w = a \ ab \]
... Are Also Sufficient!

\[ L = \{ a, ab, bba, ba, \ldots \} \]

\[ aabab, ababaabba, ababaabbaba, \ldots \}\]

\[ w = a \ ab \]
...Are Also Sufficient!

\[ L = \{ a, ab, bba, ba, \ldots \} \]

\[ abab, ababaabba, ababaabbaba, \ldots \}\]

\[ w = abab \]
... Are Also Sufficient!

\[ L = \{ a, ab, bba, ba, \ldots \} \]

\[ abab, ababaabba, ababaabbbaba, \ldots \} \]

\[ w = abab \]
Recurrent Words

\[ L = \{ a, ab, bba, ba, \ldots \} \]

\[ abab, ababaabba, ababaabbbabab \]

\[ w = abab \quad bba \]
... Are Also Sufficient!

\[ L = \{ a, ab, bba, ba, \ldots \} \]

\[ \text{abab, ababababa, ababaabbaba, \ldots} \]

\[ w = abab \quad bba \]
...Are Also Sufficient!

\[ L = \{a, ab, bba, ba, \ldots \} \]
\[ abab, ababaabba, ababaabbaba, \ldots \} \]

\[ w = ababaabba \]
Are Also Sufficient!

\[ L = \{ a, ab, bba, ba, \ldots \} \]

\[ \text{abab, ababaabba, ababaabbaba, \ldots} \}

\[ w = ababaabba \]
...Are Also Sufficient!

\[ L = \{a, ab, bba, ba, \ldots \}
\]
\[ abab, ababaabba, ababaabbbaba, \ldots \} \]

\[ w = ababaabba \quad ba \]
...Are Also Sufficient!

\[ L = \{ a, ab, bba, ba, \ldots \} \]

\[ abab, ababaabba, ababaabbbaba, \ldots \} \]

\[ w = ababaabba \quad ba \]
L = \{a, ab, bba, ba, \ldots
abab, ababaabba, ababaabbbaba, \ldots\}\n
w = ababaabbbaba
...Are Also Sufficient!

\[ L = \{a, ab, bba, ba, \ldots, \]
\[ \quad abab, ababaabba, ababaabbaba, \ldots\}\]
Recurrent Full Words

$L = \{ ba, a, ab, bba, \ldots \\
\hspace{1cm} abab, ababaabba, ababaabbaba, \ldots \}$
Recurrent Full Words

\[ L = \{ ba, a, ab, bba, \ldots \} \]

\[ w = \]

\[ \{ abab, ababaabba, ababaabbbaba, \ldots \} \]
Recurrent Full Words

\[ L = \{ba, a, ab, bba, \ldots \} \]

\[ \ldots abab, ababaabba, ababaabbaba, \ldots \} \]
Recurrent Full Words

\[ L = \{ ba, a, ab, bba, \ldots \} \]

\[ abab, ababaabba, ababaabbaba, \ldots \}\]

\[ w = ba \]
Recurrent Full Words

\[ L = \{ ba, a, ab, bba, \ldots \} \]
\[ \quad abab, ababaabba, ababaabbbaba, \ldots \} \]
\[ w = ba \]
Recurrent Full Words

\[ L = \{ ba, a, ab, bba, \ldots \} \]
\[ abab, ababaabba, ababaabbbaba, \ldots \} \]

\[ w = ba \ a \]
Recurrent Full Words

\[ L = \{ ba, a, ab, bba, \ldots \}
\]

\[ abab, ababaabba, ababaabbbaba, \ldots \}\]

\[ w = ba \ a \]
Recurrent Full Words

\[ L = \{ ba, a, ab, bba, \ldots \} \]

\[ abab, ababaabba, ababaabbaba, \ldots \} \]

\[ w = baba \]
Recurrent Full Words

\[ L = \{ ba, a, ab, bba, \ldots \} \]

\[ abab, ababaabba, ababaabbbaba, \ldots \} \]

\[ w = baba \]
Recurrent Full Words

\[ L = \{ ba, a, ab, bba, \ldots \} \]

\[ abab, ababaabba, ababaabbaba, \ldots \} \]

\[ w = baba \quad ab \]
Recurrent Full Words

\[ L = \{ ba, a, ab, bba, \ldots \] 
\[ abab, a babaabba, ababaabbbaba, \ldots \} \]

\[ w = baba \quad ab \]
Recurrent Full Words

\[ L = \{ ba, a, ab, bba, \ldots \\ abab, ababaabba, ababaabbaba, \ldots \} \]

\[ w = babaab \]
Recurrent Full Words

\[ L = \{ ba, a, ab, bba, \ldots \}
\]

\[ abab, ababaabba, ababaabbaba, \ldots \}

\[ w = babaab \]
Recurrent Full Words

\[ L = \{ ba, a, ab, bba, \ldots \\
abab, ababaabba, ababaabbbaba, \ldots \} \]

\[ w = babaab \]
Recurrent Full Words

\[ L = \{ ba, a, ab, bba, \ldots \} \]

abab, ababaabba, ababaabbbaba, \ldots \}

\[ w = babaab \ldots \]
Proposition

Let \( w \) be a recurrent partial word. There exists a completion \( \hat{w} \) of \( w \) with \( \text{Sub}(\hat{w}) = \text{Sub}(w) \).
Recurrent Full Words... and Partial Words!

Proposition

Let $w$ be a recurrent partial word. There exists a completion $\hat{w}$ of $w$ with $\text{Sub}(\hat{w}) = \text{Sub}(w)$.

Corollary

If $L$ is representable by a recurrent partial word, it is also representable by a recurrent full word.
Proposition

Let $v, w$ be partial words such that $\text{Sub}(w) = \text{Sub}(v)$. Then for each $n \geq 0$, $\sigma_n(w)$ is recurrent if and only if $\sigma_n(v)$ is.
Correlation of Ultimate Recurrence

**Proposition**

Let $v, w$ be partial words such that $\text{Sub}(w) = \text{Sub}(v)$. Then for each $n \geq 0$, $\sigma_n(w)$ is recurrent if and only if $\sigma_n(v)$ is.

**Corollary**

No language can be the subword language of both a recurrent word and a nonrecurrent word.
Correlation of Ultimate Recurrence

Proof of $n = 0$ case.
Correlation of Ultimate Recurrence

Proof of \( n = 0 \) case.

Let \( \text{Sub}(w) = \text{Sub}(v) \) and let \( w = \sigma_0(w) \) be recurrent.
Proof of $n = 0$ case.

Let $\text{Sub}(w) = \text{Sub}(v)$ and let $w = \sigma_0(w)$ be recurrent.

\[ w = w_0 w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_9 w_{10} w_{11} \cdots \]

\[ v = v_0 v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8 v_9 v_{10} v_{11} v_{12} v_{13} \cdots \]
Correlation of Ultimate Recurrence

Proof of \( n = 0 \) case.

Let \( \text{Sub}(w) = \text{Sub}(v) \) and let \( w = \sigma_0(w) \) be recurrent.

\[
\begin{align*}
\sigma_0(w) &= w_0 w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_9 w_{10} w_{11} \cdots \\
v &= v_0 v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8 v_9 v_{10} v_{11} v_{12} v_{13} \cdots
\end{align*}
\]
Correlation of Ultimate Recurrence

Proof of $n = 0$ case.

Let $\text{Sub}(w) = \text{Sub}(v)$ and let $w = \sigma_0(w)$ be recurrent.

\[
    w = w_0 w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_9 w_{10} w_{11} \cdots
\]

\[
    v = v_0 v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8 v_9 v_{10} v_{11} v_{12} v_{13} \cdots
\]
Correlation of Ultimate Recurrence

Proof of $n = 0$ case.

Let $\text{Sub}(w) = \text{Sub}(v)$ and let $w = \sigma_0(w)$ be recurrent.

\[ w = w_0 w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_9 w_{10} w_{11} \cdots \]

\[ v = v_0 v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8 v_9 v_{10} v_{11} v_{12} v_{13} \cdots \]
Correlation of Ultimate Recurrence

Proof of $n = 0$ case.

Let $\text{Sub}(w) = \text{Sub}(v)$ and let $w = \sigma_0(w)$ be recurrent.

\[
    w = w_0 w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_9 w_{10} w_{11} \cdots
\]

\[
    v = v_0 v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8 v_9 v_{10} v_{11} v_{12} v_{13} \cdots
\]
Correlation of Ultimate Recurrence

**Proof of \( n = 0 \) case.**

Let \( \text{Sub}(w) = \text{Sub}(v) \) and let \( w = \sigma_0(w) \) be recurrent.

\[
w = w_0 w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_9 w_{10} w_{11} \cdots
\]

\[
v = v_0 v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8 v_9 v_{10} v_{11} v_{12} v_{13} \cdots
\]
Proof of $n = 0$ case.

Let $\text{Sub}(w) = \text{Sub}(v)$ and let $w = \sigma_0(w)$ be recurrent.

$$w = w_0 w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_9 w_{10} w_{11} \cdots$$

$$v = v_0 v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8 v_9 v_{10} v_{11} v_{12} v_{13} \cdots$$
Nonultimate Recurrence Implies Uniqueness

Theorem

Let $L = \text{Sub}(w) = \text{Sub}(v)$, both not ultimately recurrent. Then $w = v$. 
Nonultimate Recurrence Implies Uniqueness

**Theorem**

Let \( L = \text{Sub}(w) = \text{Sub}(v) \), both not ultimately recurrent. Then \( w = v \).

**Proof (Sketch.)**

Step 1 If \( w \uparrow v \) and \( \text{Sub}(w) = \text{Sub}(v) \), but \( w \neq v \), then \( w, v \) are both ultimately recurrent.

Step 2 If \( \text{Sub}(w) = \text{Sub}(v) \) and \( w \) is not compatible with \( v \), then \( w, v \) are both recurrent.
Nonultimate Recurrence Implies Uniqueness

Theorem

Let \( L = \text{Sub}(w) = \text{Sub}(v) \), both not ultimately recurrent. Then \( w = v \).

Proof (Sketch.)

**Step 1** If \( w \uparrow v \) and \( \text{Sub}(w) = \text{Sub}(v) \), but \( w \neq v \), then \( w, v \) are both ultimately recurrent.
Nonultimate Recurrence Implies Uniqueness

**Theorem**

Let \( L = \text{Sub}(w) = \text{Sub}(v) \), both not ultimately recurrent. Then \( w = v \).

**Proof (Sketch.)**

- **Step 1** If \( w \uparrow v \) and \( \text{Sub}(w) = \text{Sub}(v) \), but \( w \neq v \), then \( w, v \) are both ultimately recurrent.
- **Step 2** If \( \text{Sub}(w) = \text{Sub}(v) \) and \( w \) is not compatible with \( v \), then \( w, v \) are both recurrent.
Nonrecurrence Implies Ultimate Uniqueness

Theorem

Let $L = \text{Sub}(w) = \text{Sub}(v)$, both not recurrent. Then $\sigma_n(w) = \sigma_n(v)$ for some $n \geq 0$. 

Proof (Sketch.)

Step 1 If $w \uparrow v$ and $\text{Sub}(w) = \text{Sub}(v)$, but $w \neq v$ infinitely often, then $w, v$ are both recurrent.

Step 2 If $\text{Sub}(w) = \text{Sub}(v)$ and $w$ is not compatible with $v$, then $w, v$ are both recurrent.

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Partial Subword Languages
Nonrecurrence Implies Ultimate Uniqueness

Theorem

Let $L = \text{Sub}(w) = \text{Sub}(v)$, both not recurrent. Then $\sigma_n(w) = \sigma_n(v)$ for some $n \geq 0$.

Proof (Sketch.)
Nonrecurrence Implies Ultimate Uniqueness

Theorem

Let $L = \text{Sub}(w) = \text{Sub}(v)$, both not recurrent. Then $\sigma_n(w) = \sigma_n(v)$ for some $n \geq 0$.

Proof (Sketch.)

Step 1 If $w \uparrow v$ and $\text{Sub}(w) = \text{Sub}(v)$, but $w \neq v$ infinitely often, then $w, v$ are both recurrent.
Nonrecurrence Implies Ultimate Uniqueness

Theorem

Let \( L = \text{Sub}(w) = \text{Sub}(v) \), both not recurrent. Then \( \sigma_n(w) = \sigma_n(v) \) for some \( n \geq 0 \).

Proof (Sketch.)

Step 1 If \( w \uparrow v \) and \( \text{Sub}(w) = \text{Sub}(v) \), but \( w \neq v \) infinitely often, then \( w, v \) are both recurrent.

Step 2 If \( \text{Sub}(w) = \text{Sub}(v) \) and \( w \) is not compatible with \( v \), then \( w, v \) are both recurrent.
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Questions?