Lecture 1 : 1.1, 1.2, 1.3

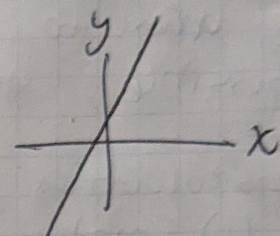
- Introduction
- Discuss syllabus & course schedule

Section 1.1 Systems of Linear Equations

Recall a line in $\mathbb{R}^2$ can be expressed as set of points $(x,y) \in \mathbb{R}^2$ such that ($\mathbb{R}$ real numbers)

$ax + by = c$ for fixed $a, b, c \in \mathbb{R}$.

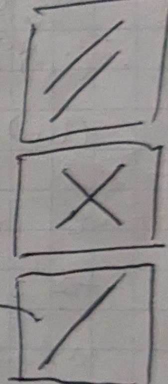
e.g. $b = 1$, $a = 2$, $c = 1$ $\quad y = -2x + 1$



Given two lines in $\mathbb{R}^2$,

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

do they intersect? If so, where? Three cases



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This is an example of a linear system of equations.

[Do similar for 3 planes in $\mathbb{R}^3$.]

One goal of course: • solve such systems

another: • understand properties & characteristics of such systems

Arise in practice (science, engineering, CS, ...)

Calculus: understand nonlinear phenomena by linearization

Def: A linear system of equations is a set of equations of the form

$$(*)\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_n \end{cases}$$

for fixed

$a_{ij} \in \mathbb{R}$ $\quad i = 1, \dots, m$

$\qquad\qquad\quad j = 1, \dots, n$

$b_j \in \mathbb{R}$ $\quad$ "

m equations, n unknowns $(x_1, \dots, x_n)$

Given a linear system of equations (*),

(aside: often, just say system of equations understanding its linear, eg. not $x^2+y^2=1$)

we say its solution set is the set of values $(x_1, \dots, x_n) \in \mathbb{R}^n$ such that ~~(*) is~~ each equation of (*) is satisfied.

We aim to understand the solution set and, when possible, to actually find solutions.

The solution set of (*) ~~is~~ either

(i) is Empty, i.e., no solutions

(ii) has Exactly one solution

(iii) has infinitely many solutions

(*) is consistent if 1 or ∞'ly many solutions.

(we will prove this later in the course, just note it for now. Another goal of course: gain familiarity w/ proofs! Explain).

ex/ $2x + y = 3$

$x - 2y = 4$

solve multiply first by two and add:

$$\begin{aligned} 4x + 2y &= 6 \\ + \quad x - 2y &= 4 \\ \hline 5x &= 10 \Rightarrow x = 2 \end{aligned}$$

plug back in to one of them $\Rightarrow y = -1$

solution is $(x, y) = (2, -1)$. Check.

~~ex/ $x + y + 2z = 1$~~

~~$3x + y + z = 2 \rightarrow 2x - z = 1$~~

~~$-x - y - 2z = 1$~~

Matrix Notation

- Given a sys. of eqns. (*), we refer to the array $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ as the coefficient matrix

and $\begin{bmatrix} & " & b_1 \\ & & \vdots \\ & & b_n \end{bmatrix}$ as the augmented matrix

e.g. previous example $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 4 \end{bmatrix}$.

ex/
$$\begin{aligned} x + y &= 1 \\ 2y + z &= 0 \\ \cancel{x} \quad -z &= 2 \end{aligned} \qquad \left[\begin{array}{ccc:c} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & -1 & 2 \end{array}\right]$$

Solve: ~~multiply 3rd row by and add to 1st row~~

~~⇒ y + z = -1~~
~~2y + z = 0~~
~~x − z = 2~~

multiply first row by −1 and add to 3rd

$-y - z = 1$

we have these eqns
$$\begin{aligned} x+y&=1 \\ 2y+z&=0 \\ -y-z&=1 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right]$$

multiply 3rd row by 2, add to 2nd
$$\begin{aligned} x+y &= 1 \\ 2y+z &= 0 \\ -z &= 2 \end{aligned}$$

⇒ add 2nd & 3rd $2y = 2$

$$\begin{aligned} x+y&=1 \\ 2y&=2 \\ -z&=2 \end{aligned} \longrightarrow \begin{aligned} x+y&=1 \\ y&=1 \\ z&=-2 \end{aligned} \rightarrow \begin{aligned} x&=0 \\ y&=1 \\ z&=-2 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

subtract 2nd from 1st

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$
echelon.

Elementary Row Operations:

Operations for trying to solve (*) can be understood as 3 basic operations on its augmented matrix

(i) Replacement: Replace one row by the sum of itself and a multiple of another row

(ii) Interchange: two rows

(iii) scaling: multiply any row by nonzero constant.

* Call two augmented matrices row equiv. if $\exists$ sequence of EROs taking one $\to$ other.

Observe: these operations are reversible. Furthermore, operations do not affect the solution set of a linear system

$\Rightarrow$ Theorem:

If the augmented matrices are row equiv., then the two systems have the same solution set $\square$

Section 1.2: Row Reduction & Echelon Forms

· Discuss previous example computation.

Triangular.

We have reduced the augmented matrix to a special form.

Def: A rect. matrix is in echelon (or row echelon) form if it satisfies:

(i) All nonzero rows are above any rows of zeros

(ii) Each leading entry of a row is in a column to the right of the leading entry of the row above it

(iii) All entries in a column below a leading entry are zeros

· reduced ref (rref) if: leading entry in each nonzero row is 1 & leading 1 is only nonzero entry in column.

ex/ $x+y+2z=2$
$x-y=0$
$x+y+z=1$

pivot

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \rightarrow \overbrace{\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -2 & -2 & -2 \\ 0 & 0 & -1 & -1 \end{bmatrix}}^{\text{ref}}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow (x,y,z) = (0,0,1)$$

ex/ $x-y=1$
$x+y+u+v=0$
$y-u-v=\cancel{\ }1$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & -1 \\ 0 & 1 & -1 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & -1 \\ 0 & 0 & -3 & -3 & 3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} \Rightarrow \begin{aligned} x &= 1 \\ y &= 0 \\ v &= -u-1 \end{aligned}$$

free variable.

solution set

$$\left\{ (x,y,u,v) : \begin{aligned} x &= 1 \\ y &= 0 \\ v &= -u-1 \\ u &\in \mathbb{R} \end{aligned} \right\}$$

__Theorem__:

- A linear system (has solution) is consistent if and only if the last column of the matrix is not a pivot column, i.e., an echelon form of the augmented matrix has no row of the form $[0 \cdots 0 \ b]$ $b \neq 0$ since this equation reads $0 = b$ ⚡.

- If a linear system is consistent, the solution contains either
  (i) a unique solution, when there are no free variables
  (ii) infinitely many solutions, when there is at least one free variable.

__section 1.3__: __Vector Equations__

$\mathbb{R}^n$ = the set of n-tuples of real numbers

A vector in $\mathbb{R}^n$ is a matrix with one column with n entries, i.e., of the form

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad x_1, \ldots, x_n \in \mathbb{R}$$

Vectors in $\mathbb{R}^n$ ~~can~~ are equipped with two fundamental operations:

scalar multiplication: given $c \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$,

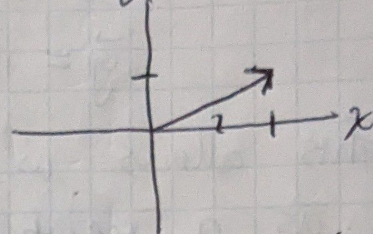
$$c\vec{x} = c\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix}$$

vector addition: given $\vec{x}$ and $\vec{y} \in \mathbb{R}^n$,

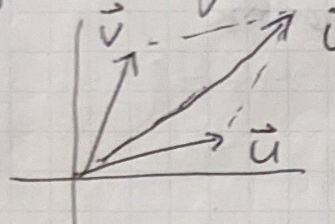
$\vec{x} + \vec{y} = \ldots$

ex/ Vectors in $\mathbb{R}^2$

- We visualize vectors $\in \mathbb{R}^2$ as arrows in the plane, e.g. $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow$



- Parallelogram rule: given two vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$, their sum is the 4^{th} vertex of the parallelogram with vertices $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{u}, \vec{v}$



e.g. $\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- Scaling: $c\vec{x}$

Linear Combinations:

Vectors in $\mathbb{R}^n$ have following algebraic props.
for any $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$

(i) Commutativity $\vec{u}+\vec{v} = \vec{v}+\vec{u}$

(ii) Associativity $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

(iii) Additive Identity $\vec{u}+\vec{0} = \vec{0}+\vec{u} = \vec{u}$

(iv) Additive inverse $\vec{u}+(-\vec{u}) = \vec{0}$ $\quad -\vec{u} = (-1)\vec{u}$

(v) Distributivity $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$

(vi & vii) Scalar compatability

$(c+d)\vec{u} = c\vec{u} + d\vec{u}$

$c(d\vec{u}) = (cd)\vec{u}$

(viii) Scalar multiplicative identity

$1\vec{u} = \vec{u}$.

A combination of scalar multiples and vector additions of vectors in $\mathbb{R}^n$ is called a linear combination

$$\vec{y} = c_1 \vec{v}_1 + \dots + \underset{\text{coefficients}}{c_p} \vec{v}_p$$

- This allows us to think of a linear system (**) as a vector equation

$$x_1 \vec{a}_1 + \dots + x_n \vec{a}_n = \vec{b}$$ in unknown coefficients $x_1, \dots, x_n$.

Corresponding augmented matrix

$$\begin{bmatrix} | & & | & | \\ \vec{a}_1 & \cdots & \vec{a}_n & \vec{b} \\ | & & | & | \end{bmatrix}.$$

Def: Given $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^n$, the set of all linear combinations of $\vec{v}_1, \dots, \vec{v}_p$ is called their span.

$$\text{span}\{\vec{v}_1, \dots, \vec{v}_p\} = \left\{ c_1\vec{v}_1 + \dots + c_p\vec{v}_p : c_1, \dots, c_p \in \mathbb{R} \right\}$$

It can be visualized as a hyperplane in $\mathbb{R}^n$ containing the origin $\vec{0}$.

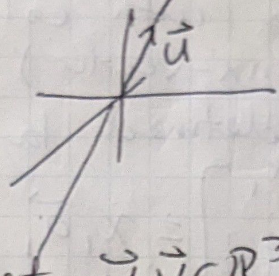
This will be useful to understand the solution set of a linear system, since

$$\vec{b} \in \text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$$

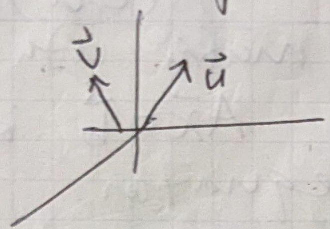
if and only if there is a solution $(x_1, \dots, x_p)$ to the linear system

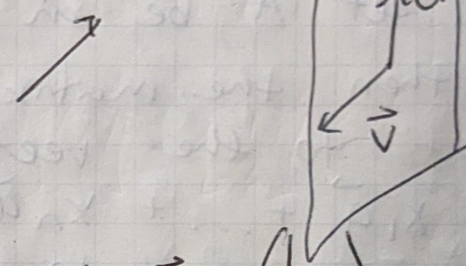
$$x_1 \vec{v}_1 + \dots + x_p \vec{v}_p = \vec{b}$$

span$\{\vec{u}\}$ let $\vec{u} \in \mathbb{R}^3$ be nonzero. Its span can be visualized as the line in $\mathbb{R}^3$ in the direction $\vec{u}$:



span$\{\vec{u}, \vec{v}\}$ let $\vec{u}, \vec{v} \in \mathbb{R}^3$ be non-colinear.
span is plane containing $\vec{u}, \vec{v}$

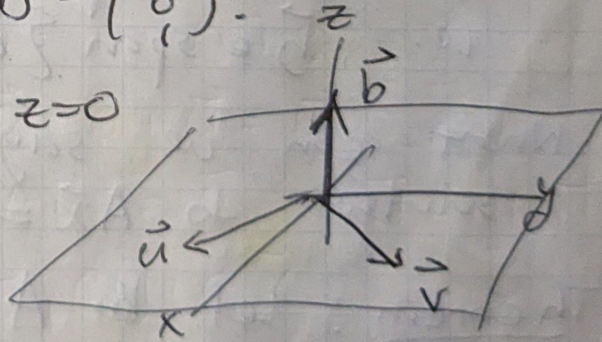




ex/ Let $\vec{u} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

What is span$\{\vec{u}, \vec{v}\}$? geometrically?
Use this to determine if there is a solution to the linear system w/ aug. matrix $[\vec{u} \;\; \vec{v} \;\; \vec{b}]$ where $\vec{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

span$\{\vec{u}, \vec{v}\}$ = xy plane at $z=0$



$\vec{b} \notin$ span$\{\vec{u}, \vec{v}\}$
$\Rightarrow$ there is no solution

$\square$