Lecture 12 - Orthogonality, Gram-Schmidt Process

Friday, July 28, 2023 9:56 AM

- HW 3 Solutions posted later today
- Practice final posted later this weekend
- Reminder to do your Student Evaluations of Teaching (SETs), open today

Def: An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that's also an orthogonal set.

Theorem:
Let $\{\vec{u}_1, \dots, \vec{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$.
Let $\vec{y} \in W$. Then, there is the unique expansion in this basis given by

$$\vec{y} = \left(\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + \dots + \left(\frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\right)\vec{u}_p .$$

proof: Since $\vec{y} \in W$ & $\{\vec{u}_1, \dots, \vec{u}_p\}$ is a basis for W, know there exists unique $c_1, \dots, c_p$ s.t.

$$\vec{y} = c_1\vec{u}_1 + \dots + c_p\vec{u}_p$$

Dot product this with $\vec{u}_j$ $(j = 1, \dots, p)$

$$\vec{y}\cdot\vec{u}_j = (c_1\vec{u}_1 + \dots + c_p\vec{u}_p)\cdot\vec{u}_j$$

$$= c_j \underbrace{\vec{u}_j\cdot\vec{u}_j}_{\|\vec{u}_j\|^2} \quad \Rightarrow c_j = \frac{\vec{y}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j}$$

$$\underbrace{c_j\, \vec{u}_j \cdot \vec{u}_j}_{\|\vec{u}_j\|^2 > 0} \quad \Rightarrow \quad c_j = \frac{\vec{y}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j}, \quad j = 1, \dots, p \qquad \square$$

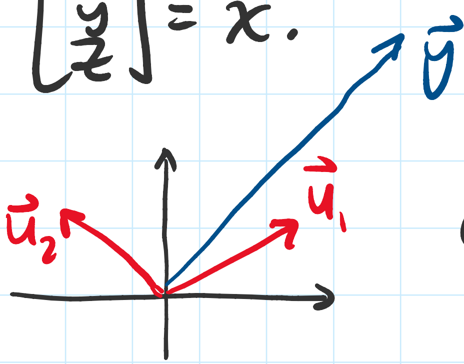
ex Let $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Verify the above formula w/ orthogonal basis $\left\{ \underset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{e}_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$

$$\frac{\vec{x}\cdot\vec{e}_1}{\vec{e}_1\cdot\vec{e}_1}\vec{e}_1 + \frac{\vec{x}\cdot\vec{e}_2}{\vec{e}_2\cdot\vec{e}_2}\vec{e}_2 + \frac{\vec{x}\cdot\vec{e}_3}{\vec{e}_3\cdot\vec{e}_3}\vec{e}_3$$

$$= \frac{x}{1}\vec{e}_1 + \frac{y}{1}\vec{e}_2 + \frac{z}{1}\vec{e}_3 = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{x}.$$

ex



$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \vec{u}_1\cdot\vec{u}_2 = 0$

$\vec{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

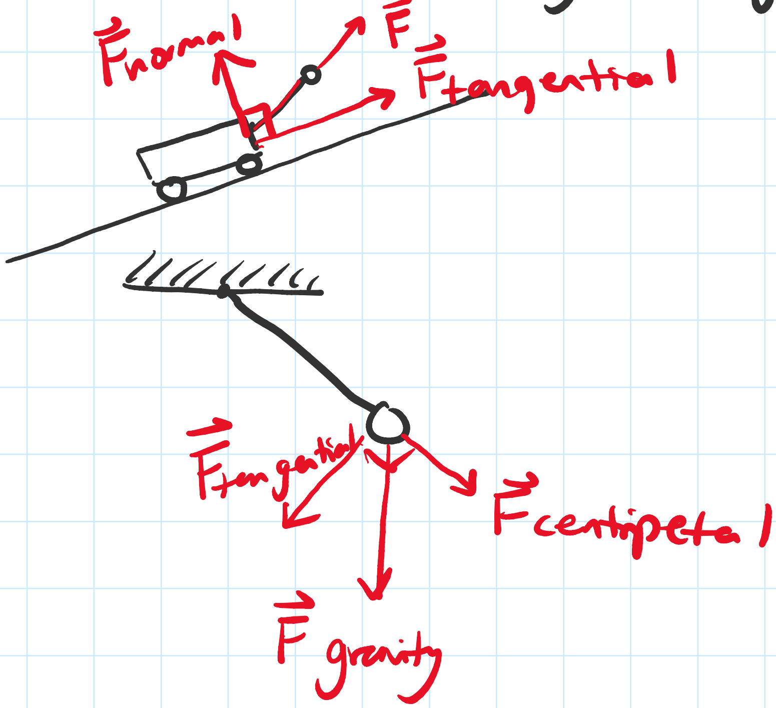
Write $\vec{y}$ as a lin comb of $\vec{u}_1, \vec{u}_2$

$$\vec{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2$$

$$= \frac{5}{2}\vec{u}_1 + \frac{1}{2}\vec{u}_2 \qquad \left(\overset{\text{check}}{\frac{5}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}} \checkmark \right)$$

$$= \frac{5}{2}u_1 + \frac{1}{2}u_2 \quad \left(\frac{5}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix} \checkmark \right)$$

ex/ Physics: free body diagrams



Def:

- A vector $\vec{u}$ is a unit vector when $\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = 1$
- An orthonormal set is an orthogonal set of unit vectors.

ex/ standard basis $\mathbb{R}^n$ is orthonormal.

An orthonormal set of vectors $\{\vec{u}_1, \dots, \vec{u}_p\}$ satisfies:

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Given $\vec{y} \in W = \text{span}\{\vec{u}_1, \dots, \vec{u}_p\}$ subspace of $\mathbb{R}^n$

Given $\vec{y} \in W = \text{span}\{\vec{u}_1, \dots, \vec{u}_p\}$ subspace of $\mathbb{R}^n$

$$\vec{y} = (\vec{y}\cdot\vec{u}_1)\vec{u}_1 + \dots + (\vec{y}\cdot\vec{u}_p)\vec{u}_p$$

Let $\{\vec{u}_1, \dots, \vec{u}_n\}$ be an orthonormal basis for $\mathbb{R}^n$.

Let
$U = [\vec{u}_1 \cdots \vec{u}_n]$ U in invertible, claim $U^{-1} = U^T$

Observe

$$U^T U = \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix} [\vec{u}_1 \cdots \vec{u}_n]$$

$$= \begin{bmatrix} \vec{u}_1^T\vec{u}_1 & \vec{u}_1^T\vec{u}_2 & \cdots & \vec{u}_1^T\vec{u}_n \\ \vdots & \vec{u}_2^T\vec{u}_2 & & \vdots \\ & & \ddots & \\ \vec{u}_n^T\vec{u}_1 & \cdots & & \vec{u}_n^T\vec{u}_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = I \Rightarrow U^T = U^{-1}.$$

such $n \times n$ matrices w/ orthonormal columns (or equivalently whose transpose is its inverse) are called orthogonal matrices.
(bad nomenclature should be orthonormal)

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(bad nomenclature ↑ should be orthonormal)

consider linear system $U\vec{x} = \vec{y}$
where U is an orthogonal matrix.

$$\Rightarrow \vec{x} = U^T\vec{y}$$

$$U\vec{x} = \vec{y} \iff \vec{y} = x_1\vec{u}_1 + \ldots + x_n\vec{u}_n$$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \vec{u}_1^{\,T} \\ \vdots \\ \vec{u}_n^{\,T} \end{bmatrix}\vec{y} = \begin{bmatrix} \vec{u}_1^{\,T}\vec{y} \\ \vdots \\ \vec{u}_n^{\,T}\vec{y} \end{bmatrix} = \begin{bmatrix} \vec{y}\cdot\vec{u}_1 \\ \vdots \\ \vec{y}\cdot\vec{u}_n \end{bmatrix}$$

Theorem: [The Spectral Theorem]

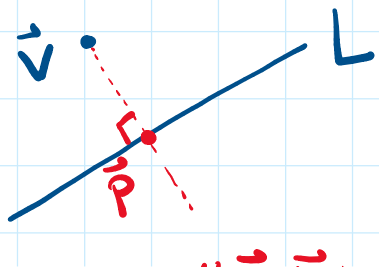
a matrix A is orthogonally diagonalizable,
i.e., there exists an orthogonal matrix U
and a diagonal matrix D s.t.

$$A = UDU^{-1} = UDU^T$$

if and only if A is symmetric ($A = A^T$) $\square$

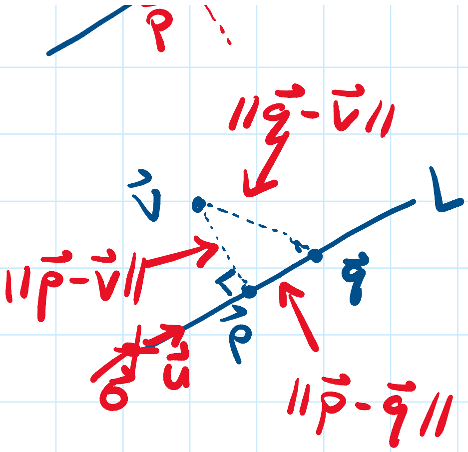
[Orthogonal Projections & Decompositions]

ex/ Consider



What is the point on L closest to $\vec{v}$?

$\vec{p}$ is the point on L closest



$\vec{p}$ is the point on L closest to $\vec{v}$.

$\vec{q} \neq \vec{p}$, $\vec{q}$ on L

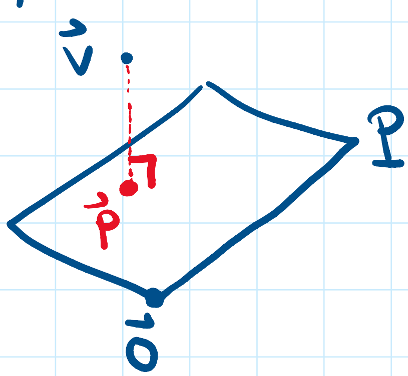
$$\|\vec{q}-\vec{v}\|^2 = \|\vec{p}-\vec{v}\|^2 + \underbrace{\|\vec{p}-\vec{q}\|^2}_{>0}$$

$$> \|\vec{p}-\vec{v}\|^2$$

i.e. $\|\vec{p}-\vec{v}\| < \|\vec{q}-\vec{v}\|$ for all $\vec{q}$ on L not equal to $\vec{p}$.

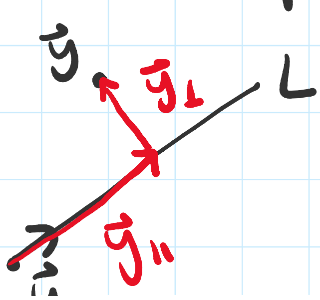
we call $\vec{p}$ the orthogonal projection of $\vec{v}$ onto $L = \text{span}\{\vec{u}\}$. $\text{proj}_L \vec{v}$.

<u>ex</u> plane P in $\mathbb{R}^3$



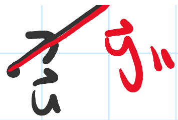
$\vec{p} = \text{proj}_P \vec{v}$

Going back to first example; let $\vec{u} \in \mathbb{R}^n$, $\vec{u} \neq \vec{0}$
let $L = \text{span}\{\vec{u}\}$. Let $\vec{y} \in \mathbb{R}^n$.



Can decompose

$$\vec{y} = \vec{y}_\parallel + \vec{y}_\perp$$



$$\vec{y} = \underset{\in L}{\vec{y}_{\parallel}} + \underset{\in L^{\perp}}{\vec{y}_{\perp}}$$

know $\vec{y}_{\parallel} = \alpha \vec{u}$ for some $\alpha \in \mathbb{R}$. Enforce

$\vec{y}_{\perp} = \vec{y} - \vec{y}_{\parallel} \in L^{\perp}$

$$0 \overset{!}{=} \vec{y}_{\perp} \cdot \vec{u} = (\vec{y} - \vec{y}_{\parallel}) \cdot \vec{u} = \vec{y} \cdot \vec{u} - \vec{y}_{\parallel} \cdot \vec{u}$$

$$= \vec{y} \cdot \vec{u} - \alpha \underbrace{\vec{u} \cdot \vec{u}}_{\neq 0} \Rightarrow \alpha = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$$

$$\Rightarrow \vec{y}_{\parallel} = \left(\frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \right) \vec{u}$$

$$\vec{y}_{\perp} = \vec{y} - \vec{y}_{\parallel}$$

<u>Thm</u>: [Orthogonal Decomposition Theorem]
Let W be a subspace of $\mathbb{R}^n$.
Then, for any $\vec{y} \in \mathbb{R}^n$, there is a unique decomposition

$$\vec{y} = \vec{y}_{\parallel} + \vec{y}_{\perp} \quad \text{where} \quad \begin{array}{l} \vec{y}_{\parallel} \in W \\ \vec{y}_{\perp} \in W^{\perp} \end{array}$$

Furthermore, assume $W \neq \{\vec{0}\}$; let $\{\vec{u}_1, \dots, \vec{u}_p\}$ be an orthogonal basis for W. Then,

$$\vec{y}_{\parallel} = \left(\frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \right) \vec{u}_1 + \dots + \left(\frac{\vec{y} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p} \right) \vec{u}_p = \text{proj}_W \vec{y}$$

$$\vec{y}_{\parallel} = \left(\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + \dots + \left(\frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\right)\vec{u}_p = \text{proj}_W \vec{y}$$

$$\vec{y}_{\perp} = \vec{y} - \vec{y}_{\parallel}.$$

<u>proof</u>: clearly $\vec{y}_{\parallel} \in W$.

$\vec{y}_{\perp} = \vec{y} - \vec{y}_{\parallel}$ check $\in W^{\perp}$.

$\Leftrightarrow$ check $\vec{y}_{\perp}\cdot\vec{u}_j = 0 \quad j = 1, \dots, p$

$$\vec{y}_{\perp}\cdot\vec{u}_j = (\vec{y} - \vec{y}_{\parallel})\cdot\vec{u}_j$$

$$= \vec{y}\cdot\vec{u}_j - \frac{\vec{y}\cdot\vec{u}_j}{\cancel{\vec{u}_j\cdot\vec{u}_j}}\cancel{\vec{u}_j\cdot\vec{u}_j} = 0, \quad j = 1, \dots, p$$

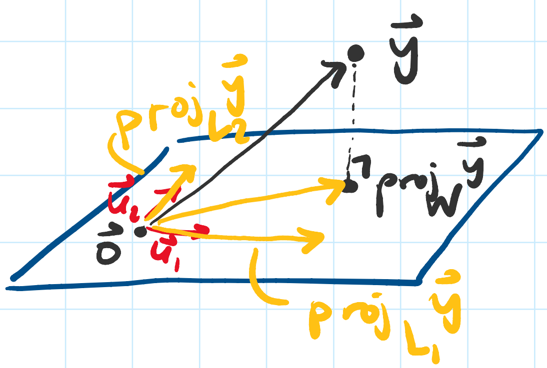
$$\Rightarrow \vec{y}_{\perp} \in W^{\perp}$$

uniqueness

$$\underset{W}{\underset{\cap}{\vec{y}_{\parallel}}} + \underset{W^{\perp}}{\underset{\cap}{\vec{y}_{\perp}}} = \vec{y} = \underset{W}{\underset{\cap}{\vec{z}_{\parallel}}} + \underset{W^{\perp}}{\underset{\cap}{\vec{z}_{\perp}}}$$

$$W \ni \vec{y}_{\parallel} - \vec{z}_{\parallel} \underset{\vec{0}}{\underset{\parallel}{=}} \vec{z}_{\perp} - \vec{y}_{\perp} \in W^{\perp}$$

$$\Rightarrow \vec{y}_{\parallel} = \vec{z}_{\parallel} \quad \& \quad \vec{y}_{\perp} = \vec{z}_{\perp} \qquad \square$$



$W = \text{span}\{\vec{u}_1, \vec{u}_2\}$

$L_1 = \text{span}\{\vec{u}_1\}$ $\quad L_2 = \text{span}\{\vec{u}_2\}$

$\text{proj}_{L_1} y \qquad L_1 = \text{span}\{\vec{u}_1\} \quad L_2 = \text{span}\{\vec{u}_2\}$

$$\text{proj}_W \vec{y} = \left(\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + \left(\frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\right)\vec{u}_2$$

$$= \text{proj}_{L_1}\vec{y} + \text{proj}_{L_2}\vec{y}$$

ex/ let $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

Let $W = \text{span}\{\vec{u}_1, \vec{u}_2\}$. Let $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Find $\text{proj}_W \vec{y}$ and the decomposition

$$\vec{y} = \underset{W}{\vec{y}_{\parallel}} + \underset{W^{\perp}}{\vec{y}_{\perp}}$$

Check $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis for W

$\vec{u}_1\cdot\vec{u}_2 = -2+2 = 0$.

$$\vec{y}_{\parallel} = \text{proj}_W \vec{y} = \left(\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + \left(\frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\right)\vec{u}_2$$

$$= \frac{2}{6}\vec{u}_1 + \frac{3}{5}\vec{u}_2 = \frac{2}{6}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \frac{3}{5}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{y}_{\perp} = \vec{y} - \vec{y}_{\parallel} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{6}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - \frac{3}{5}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

check $\vec{y}_{\perp}\cdot\vec{u}_1 = 0 = \vec{y}_{\perp}\cdot\vec{u}_2$

Theorem [Best Approximation]

- Let W be a subspace of $\mathbb{R}^n$. Let $\vec{y} \in \mathbb{R}^n$.
- Then, the orthogonal projection of $\vec{y}$ onto W,

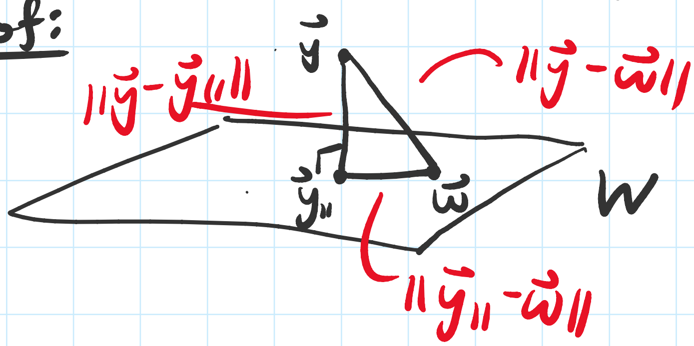
  $\vec{y}_{\parallel} = \text{proj}_W \vec{y}$, is the closest vector in

  W to $\vec{y}$, i.e.

  for any $\vec{w} \in W$ not equal to $\vec{y}_{\parallel}$,

$$\|\vec{y} - \vec{y}_{\parallel}\| < \|\vec{y} - \vec{w}\|.$$

proof:



Pythagorean thm:

$$\vec{y} - \vec{w} = \underbrace{(\vec{y} - \vec{y}_{\parallel})}_{\substack{= \vec{y}_{\perp} \\ \in W^{\perp}}} + \underbrace{(\vec{y}_{\parallel} - \vec{w})}_{\in W}$$

$$\Rightarrow \|\vec{y} - \vec{w}\|^2 = \|\vec{y} - \vec{y}_{\parallel}\|^2 + \underbrace{\|\vec{y}_{\parallel} - \vec{w}\|^2}_{>0}$$

$$> \|\vec{y} - \vec{y}_{\parallel}\|^2$$

$\square$

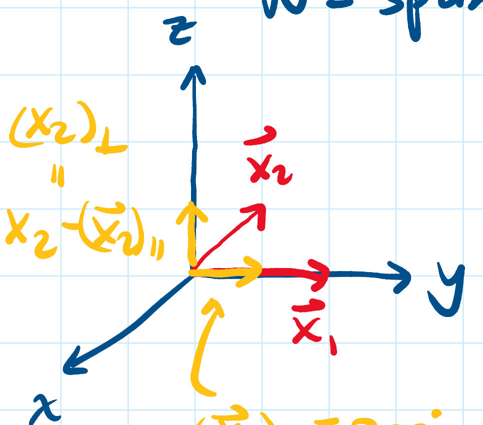
The Gram-Schmidt Process:

The Gram-Schmidt Process:

Given a basis for a subspace W of $\mathbb{R}^n$, how can we turn it into an orthogonal basis?

ex Consider $\vec{x}_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Construct orthogonal basis for $W = \text{span}\{\vec{x}_1, \vec{x}_2\}$, say $\{\vec{v}_1, \vec{v}_2\}$



$W = \text{span}\{\vec{x}_1, \vec{x}_2\}$ is the yz plane $x = 0$

$(\vec{x}_2)_{\|} = \text{proj}_{W_1} \vec{x}_2 \qquad W_1 = \text{span}\{\vec{x}_1\}$

Take

$\vec{v}_1 = \vec{x}_1$. $\quad W_1 = \text{span}\{\vec{x}_1\} = \text{span}\{\vec{v}_2\}$.

$\vec{v}_2 = \vec{x}_2 - \text{proj}_{W_1} \vec{x}_2$

Claim: $\{\vec{v}_1, \vec{v}_2\}$ is a basis for W.

"it suffices to show $\{\vec{v}_1, \vec{v}_2\}$ are lin. ind (since they're both in W)

$\vec{v}_1 \cdot \vec{v}_2 = 0$ orthogonal $\quad \vec{v}_1 = \vec{x}_1 \neq 0$

[since they're both in W]

$\vec{v}_1 \cdot \vec{v}_2 = 0$ orthogonal $\quad \vec{v}_1 = \vec{x}_1 \neq 0$

show $\vec{v}_2 \neq \vec{0}$. If $\vec{v}_2 = \vec{0}$, $\vec{x}_2 = \text{proj}_{W_1} \vec{x}_2$

$\Rightarrow \vec{x}_2 \in W_1 \Rightarrow \vec{x}_2 \in \text{span}\{\vec{x}_1\}$

$\Rightarrow$ contradiction. $\square$

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - (\vec{x}_2)_{\parallel}$$

$$= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

<u>Thm</u>: [The Gram-Schmidt Process]

Let $\{\vec{x}_1, \dots, \vec{x}_p\}$ be a basis for a subspace W of $\mathbb{R}^n$.

Let:

$\vec{v}_1 = \vec{x}_1 \qquad W_1 = \text{span}\{\vec{x}_1\} = \text{span}\{\vec{v}_1\}$

$\vec{v}_2 = \vec{x}_2 - \text{proj}_{W_1} \vec{x}_2$

$= \vec{x}_2 - \left(\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 \qquad W_2 = \text{span}\{\vec{x}_1, \vec{x}_2\}$

$= \text{span}\{\vec{v}_1, \vec{v}_2\}$.

$\vec{v}_3 = \vec{x}_3 - \text{proj}_{W_2} \vec{x}_3$

$$\vec{v}_3 = \vec{x}_3 - \text{proj}_{W_2} \vec{x}_3$$
$$= \vec{x}_3 - \left(\frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \left(\frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2$$

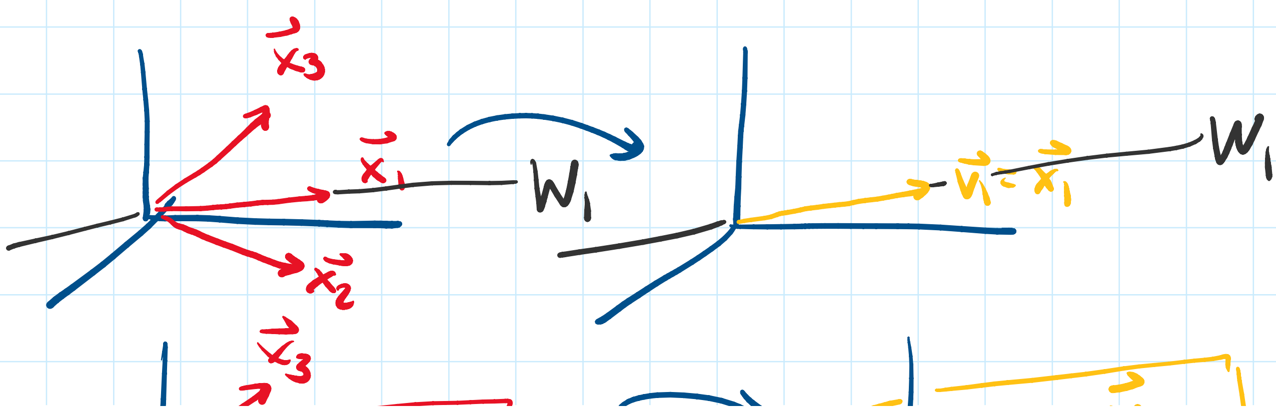
$$W_k = \text{span}\{\vec{v}_1, \dots, \vec{v}_k\}$$
where $\vec{v}_k = \vec{x}_k - \text{proj}_{W_{k-1}} \vec{x}_k$

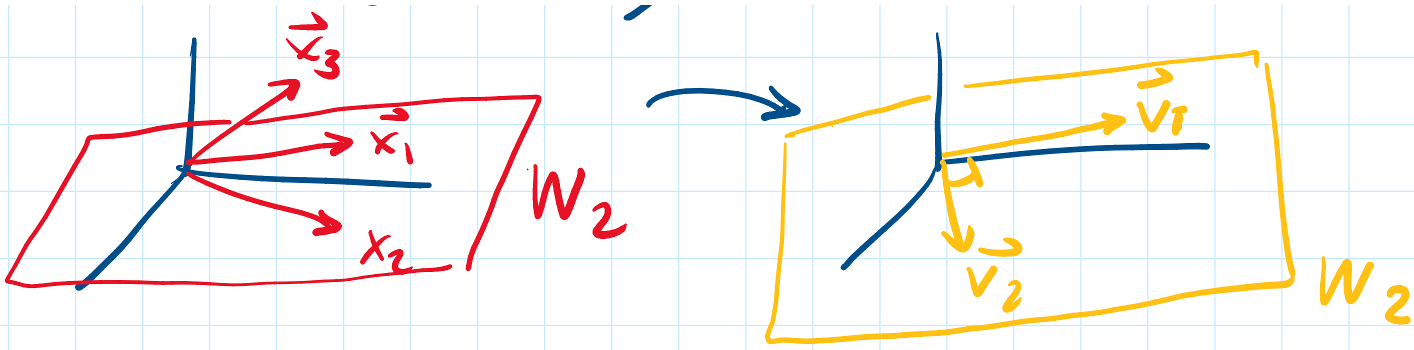
$$\vec{v}_p = \vec{x}_p - \text{proj}_{W_{p-1}} \vec{x}_p$$
$$= \vec{x}_p - \left(\frac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \dots - \left(\frac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}}\right)\vec{v}_{p-1}$$

Then, $\{\vec{v}_1, \dots, \vec{v}_p\}$ is an orthogonal basis for W w/ the additional property

$$\text{span}\{\vec{x}_1\} = \text{span}\{\vec{v}_1\}$$
$$\text{span}\{\vec{x}_1, \vec{x}_2\} = \text{span}\{\vec{v}_1, \vec{v}_2\}$$
$$\Downarrow$$
$$\text{span}\{\vec{x}_1, \dots, \vec{x}_k\} = \text{span}\{\vec{v}_1, \dots, \vec{v}_k\}$$
for $1 \le k \le p$.

$\square$





$\vec{x}_3$
$\vec{x}_1$
$\vec{x}_2$
W_2
$\vec{v}_1$
$\vec{v}_2$
W_2