Lecture 4

Monday, July 10, 2023 9:58 AM

- HW1 due Wednesday at 11:59 pm
- HW2 will be posted on Wednesday
- OH today from 1-2 pm.

<u>Prop</u>: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a lin. transf.

(i) Suppose T is injective. Then, $n \leq m$.

(ii) Suppose T is surjective. Then, $n \geq m$.

<u>proof</u>: Let $\underset{m \times n}{A}$ be the matrix assoc. w/ T

(i) T is injective $\Leftrightarrow$ cols of A are lin. indep $\Leftrightarrow$ A has # of pivots equal to the # of cols $= n$

$\Rightarrow$ # of pivots $\leq$ # of rows $= m$

$\quad\quad$ $\parallel$

$\quad\quad$ n

counter ex $\underbrace{\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}}_{n=3} \Big\} m=2$

(ii) T is surjective $\Leftrightarrow$ cols of A span $\mathbb{R}^m$ $\Leftrightarrow$ there are at least m pivots

$\geq$ # of cols $= n$

counter ex $\underbrace{\begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{bmatrix}}_{n=2} \Bigg\} m=3$

$\square$

Corollary: a bijective lin. transf. T must map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ for some n, i.e., the assoc. matrix A is square ($n \times n$).

ex/ Consider $\mathbb{1}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ "the identity function on $\mathbb{R}^n$"

$\mathbb{1}_{\mathbb{R}^n}(\vec{x}) = \vec{x}$ for all $\vec{x}$.

What's the corresponding matrix?

"the $n \times n$ identity matrix"

$$I_n = [\vec{e}_1 \cdots \vec{e}_n] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$(I_n)_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

row col

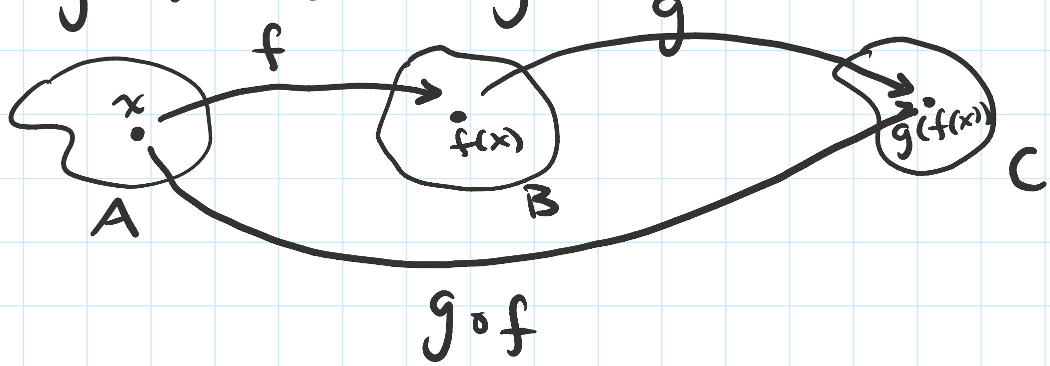
Def:

- Given two functions $f: \mathbb{R}^n \to \mathbb{R}^m$ and $g: \mathbb{R}^n \to \mathbb{R}^m$, define their sum as sum as

$$f+g : \mathbb{R}^n \to \mathbb{R}^m$$
$$(f+g)(\vec{x}) = f(\vec{x}) + g(\vec{x}).$$

- Given two functions $f : A \to B$ and $g : B \to C$, define their composition as

$$g \circ f : A \to C$$
$$g \circ f(x) = g(f(x))$$



<u>Thm:</u> Let $T : \mathbb{R}^n \to \mathbb{R}^m$, $S : \mathbb{R}^n \to \mathbb{R}^m$, $U : \mathbb{R}^m \to \mathbb{R}^p$ all be lin. transf. and let $c \in \mathbb{R}$. Then,

(i) cT is a linear transf.

(ii) $T+S$ is a linear transf.

(iii) $U \circ T$ is a linear transf.

<u>pf:</u> (i) & (ii) exercise.

(iii) $U \circ T(a\vec{x} + b\vec{y})$

$= U(T(a\vec{x} + b\vec{y}))$ T linear

$= U(aT(\vec{x}) + b\,T(\vec{y}))$ U linear

$= aU(T(\vec{x})) + bU(T(\vec{y}))$

$= a\,U \circ T(\vec{x}) + b\,U \circ T(\vec{y})$ $\square$

<u>Def:</u> [Three algebraic matrix ops]

(i) scalar-matrix multiplication

$c \in \mathbb{R}$, A $m \times n$ matrix

$$cA = c\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} := \begin{bmatrix} cA_{11} & \cdots & cA_{1n} \\ \vdots & \ddots & \vdots \\ cA_{m1} & \cdots & cA_{mn} \end{bmatrix}$$

$$(cA)_{ij} := c\,A_{ij}$$

(ii) matrix addition

A $m \times n$, B $m \times n$.

$$A + B = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} + \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mn} \end{bmatrix}$$

$$\begin{bmatrix} A_{m1} & \cdots & A_{mn} \end{bmatrix} \quad \begin{bmatrix} B_{m1} & \cdots & B_{mn} \end{bmatrix}$$

$$:= \begin{bmatrix} A_{11}+B_{11} & \cdots & A_{1n}+B_{1n} \\ \vdots & & \\ A_{m1}+B_{m1} & \cdots & A_{mn}+B_{mn} \end{bmatrix}$$

$$(A+B)_{ij} := A_{ij} + B_{ij}$$

(iii) matrix multiplication

B $p \times m$ matrix, A $m \times n$ matrix

$$BA = B[\vec{a}_1 \cdots \vec{a}_n]$$
$$= [B\vec{a}_1 \cdots B\vec{a}_n]$$

$\underset{(p \times m)}{B} \cdot \underset{(m \times n)}{A}$

$p \times n$ matrix

ex/ $B = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 4 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 0 & 2 \end{bmatrix}$$
$$\quad\;\; \vec{a}_1 \;\; \vec{a}_2$$

$$BA = \left[\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 2+3+0 & 1+9+4 \\ \cdots \end{bmatrix} = \begin{bmatrix} 5 & 14 \\ \cdots & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3+0 & 1+9+4 \\ 0+1+0 & 0+3+8 \end{bmatrix} = \begin{bmatrix} 5 & 14 \\ 1 & 11 \end{bmatrix}$$

B $p \times m$, A $m \times n$

$$(BA)_{ij} = \sum_{k=1}^{m} B_{ik} A_{kj} \qquad \begin{pmatrix} i = 1, \dots, p \\ j = 1, \dots, n \end{pmatrix}$$

ex/

$$\overset{B}{\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 4 \end{bmatrix}} \overset{A}{\begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 0 & 2 \end{bmatrix}} = \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \end{bmatrix}$$

Properties: Let A $m \times n$, B, C defined s.t. the following sums/product are defined, $d \in \mathbb{R}$. Then

(i) $A(BC) = (AB)C$

(ii) $A(B+C) = AB + AC$

(iii) $(B+C)A = BA + CA$

(iv) $d(AB) = (dA)B = A(dB)$

(v) $I_m A = A = A I_n$

Transpose: A $m \times n$. Transpose of A, denoted A^T, is the $n \times m$ matrix given by swapping the rows & cols of A.

$$(A^T)_{ij} = A_{ji}$$

ex/ $$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}^T = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix}$$

ex/ $$\begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 2 \\ 8 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 & 4 & 8 \\ 1 & 3 & 2 & 1 \end{bmatrix}$$

properties: $(A^T)^T = A$

$(A+B)^T = A^T + B^T$

$(cA)^T = cA^T$

$(BA)^T = A^T B^T$

- B $p \times m$ $\sim$ $U : \mathbb{R}^m \to \mathbb{R}^p$

  B^T $m \times p$ $\sim$ $U^* : \mathbb{R}^p \to \mathbb{R}^m$

- A $m \times n$ $\sim$ $T : \mathbb{R}^n \to \mathbb{R}^m$

  A^T $n \times m$ $\sim$ $T^* : \mathbb{R}^m \to \mathbb{R}^n$

A $m\times n$ $\sim$ $T: \mathbb{R}^n \to \mathbb{R}^m$

A^T $n\times m$ $\sim$ $T^*: \mathbb{R}^m \to \mathbb{R}^n$

(sections 2.2 & 2.3)

A $n \times n$, $\vec{x} \in \mathbb{R}^n$ unknown, $\vec{b} \in \mathbb{R}^n$ unknown

$$A\vec{x} = \vec{b}$$
$$\wr$$
$$T(\vec{x}) = \vec{b};$$ if T is invertible,
$$\vec{x} = T^{-1}(\vec{b}).$$

Theorem:

- Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a bijective lin. transf. and let A be the corresp. matrix.
- Let T^{-1} be the inverse of T, i.e.,
$$T \circ T^{-1} = \mathbb{1}_{\mathbb{R}^n}$$
$$T^{-1} \circ T = \mathbb{1}_{\mathbb{R}^n}$$
- Then, T^{-1} is also a linear transf.
- Let A^{-1} denote the matrix assoc. to T^{-1}. Then,

assoc. to T^{-1}. Then,

$$AA^{-1} = I_n = A^{-1}A.$$

proof:

Want to show:

$$T^{-1}(c_1\vec{y}_1 + c_2\vec{y}_2) = c_1 T^{-1}(\vec{y}_1) + c_2 T^{-1}(\vec{y}_2) \quad \left(\text{for all } c_1, c_2, \vec{y}_1, \vec{y}_2\right)$$

- Let $\vec{x}_1 = T^{-1}(\vec{y}_1)$ and $\vec{x}_2 = T^{-1}(\vec{y}_2)$.

$$T(\vec{x}_1) = \vec{y}_1 \qquad T(\vec{x}_2) = \vec{y}_2$$

$$c_1 T(\vec{x}_1) + c_2 T(\vec{x}_2) = c_1\vec{y}_1 + c_2\vec{y}_2$$

linearity of T

$$T(c_1\vec{x}_1 + c_2\vec{x}_2) = c_1\vec{y}_1 + c_2\vec{y}_2$$

$$\Rightarrow c_1\vec{x}_1 + c_2\vec{x}_2 = T^{-1}(c_1\vec{y}_1 + c_2\vec{y}_2)$$

$$\Rightarrow c_1 T^{-1}(\vec{y}_1) + c_2 T^{-1}(\vec{y}_2) = T^{-1}(c_1\vec{y}_1 + c_2\vec{y}_2) \quad \square$$

Def: We say a matrix $A^{(n\times n)}$ is invertible if there exists a matrix $C^{(n\times n)}$ s.t.

If there exists a matrix $C^{(n\times n)}$ s.t.

$$CA = I_n \qquad AC = I_n$$

If such a matrix C exists, we denote it as A^{-1}

(If B & C s.t. $BA = I_n = AB$,

$CA = I_n = AC$

$B = BI_n = B(AC) = (BA)C = I_n C = C$)

ex/ 2×2 matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A is invertible $\iff$ $\underbrace{ad - bc}_{\det(A)} \neq 0$ determinant of A

Can we pivot on first col?

yes only unless both $a = 0 = c$

assume one is nonzero.

WLOG that $a \neq 0$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{②-\frac{c}{a}① \to ②} \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{② - \frac{c}{a}① \to ②} \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{bmatrix}$$

$\neq 0$ for 2 pivots

$$d - \frac{c}{a}b \neq 0$$

i.e. $ad - bc \neq 0.$ $\square$

When $ad - bc \neq 0$, for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Check $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^{-1}A$)

ex/ $A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$ Is A invertible? If so, compute A^{-1}

$$\det(A) = (1)(-1) - (2)(4)$$

$$= -1 - 8 = -9 \neq 0 \Rightarrow A \text{ invertible.}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$a = 1$ $b = 4$
$c = 2$ $d = -1$

$$= \frac{-1}{9}\begin{bmatrix} -1 & -4 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{-1}{9}\begin{bmatrix} -1 & 4 \\ -2 & 1 \end{bmatrix}$$

Inverting matrices in the general $n \times n$ case.

A $n \times n$ matrix, assume its invertible

consider these n systems of equations

$$A\vec{x}_1 = \vec{e}_1, \ldots, A\vec{x}_n = \vec{e}_n$$

$$\Rightarrow \vec{x}_1 = A^{-1}\vec{e}_1, \ldots, \vec{x}_n = A^{-1}\vec{e}_n$$

<u>claim</u>: $A^{-1} = [\vec{x}_1 \cdots \vec{x}_n]$

$$= [A^{-1}\vec{e}_1 \cdots A^{-1}\vec{e}_n]$$

(see last lecture)

$$[A \mid \vec{e}_1\ \vec{e}_2 \cdots \vec{e}_n]$$

$$= [A \mid I_n] \xrightarrow[\text{reduce}]{\text{row}} [I_n \mid A^{-1}]$$

<u>exercise</u>: assuming $ad - bc \neq 0$,

check

assuming $ad-bc \neq 0$,

check

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

ex/ $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \\ 3 & 1 & 1 \end{bmatrix}$ is invertible.

$$[A \mid I_n] = \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 2 & 1 & 5 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\textcircled{3} - 3\textcircled{1} \to \textcircled{3}]{\textcircled{2} - 2\textcircled{1} \to \textcircled{2}} \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & -3 & -3 & -2 & 1 & 0 \\ 0 & -5 & -11 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\textcircled{3} - \frac{5}{3}\textcircled{2} \to \textcircled{3}]{\textcircled{1} + \frac{2}{3}\textcircled{2} \to \textcircled{1}} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -\frac{1}{3} & 2/3 & 0 \\ 0 & -3 & -3 & -2 & 1 & 0 \\ 0 & 0 & -6 & 1/3 & -5/3 & 1 \end{array}\right]$$

$$\xrightarrow[\textcircled{3} \cdot \left(\frac{-1}{6}\right)]{\textcircled{2} \cdot \left(-\frac{1}{3}\right)} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1/3 & 2/3 & 0 \\ 0 & 1 & 1 & 2/3 & -1/3 & 0 \\ 0 & 0 & 1 & -1/18 & 5/18 & -1/6 \end{array}\right]$$

$$\xrightarrow[\text{②} - \text{③} \to \text{②}]{\text{①} - 2\,\text{③} \to \text{①}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3}+\frac{1}{9} & \frac{2}{3}-\frac{5}{9} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{2}{3}+\frac{1}{18} & -\frac{1}{3}-\frac{5}{18} & \frac{1}{6} \\ 0 & 0 & 1 & -1/18 & 5/18 & -1/6 \end{array}\right]$$

$$\underbrace{\begin{matrix} -\frac{1}{3}+\frac{1}{9} & \frac{2}{3}-\frac{5}{9} & \frac{1}{3} \\ \frac{2}{3}+\frac{1}{18} & -\frac{1}{3}-\frac{5}{18} & \frac{1}{6} \\ -1/18 & 5/18 & -1/6 \end{matrix}}_{A^{-1}}$$