Lecture 8 - Determinants (cont.), Eigenvalues and Eigenvectors

Wednesday, July 19, 2023 10:00 AM

- OH today from 12 pm - 1 pm in this lecture hall & zoom
- HW2 is due tonight at 11:59 pm
- HW3 will be posted later today
- Midterm is this upcoming Friday July 21st with review session Friday morning via Zoom

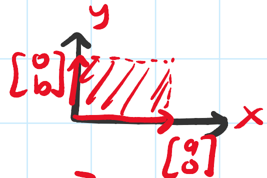
Thm: If A is a 2×2 matrix, then the area of the parallelogram determined by its columns is $|\det(A)|$.

- If A is a 3×3 matrix, then the volume of the parallelepiped determined by its columns is $|\det(A)|$.
- If A is an $n\times n$ matrix, then the n-volume of the n-parallelotope determined by its columns is $|\det(A)|$.

proof sketch:

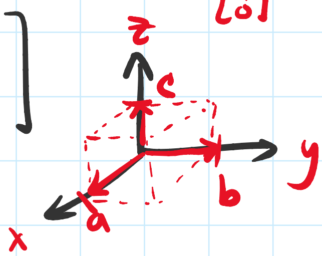
this is true for diagonal matrices

$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$



Area $= |ab| = |\det(A)|$

$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$



Vol $= |abc| = |\det(A)|$

$\square$

Let A be a 2×2 matrix (similar ideas apply $n\times n$)

Let T be the assoc. lin transf.

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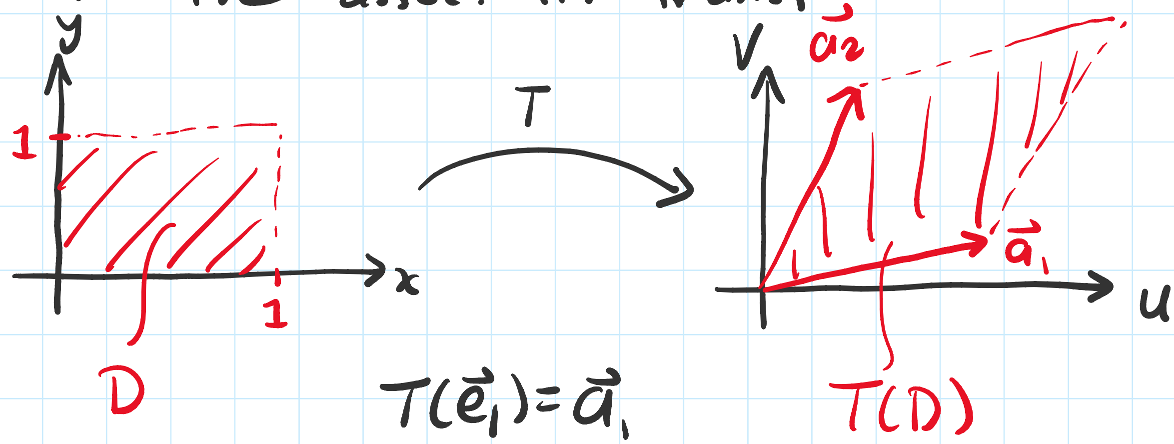
Let T be the assoc. lin transf



$$T(\vec{e}_1)=\vec{a}_1$$
$$T(\vec{e}_2)=\vec{a}_2$$

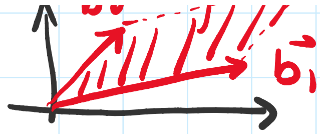
$$\text{Area}(T(D)) = |\det(A)|\cdot \text{Area}(D)$$

Thm:

- Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be a lin. transf w/ assoc. matrix A.
  Let S be a parallelogram in $\mathbb{R}^2$.
  Then, $T(S)$ is a parallelogram in $\mathbb{R}^2$ and
  $$\text{Area}(T(S)) = |\det(A)|\,\text{Area}(S)$$
- Let $Q:\mathbb{R}^3\to\mathbb{R}^3$ be a lin transf. w/ assoc. matrix B.
  Let P be a parallelepiped in $\mathbb{R}^3$.
  Then so is $Q(P)$ and
  $$\text{Vol}(Q(P)) = |\det(B)|\,\text{Vol}(P)$$
- (similar in $\mathbb{R}^n$)

Proof:

proof:



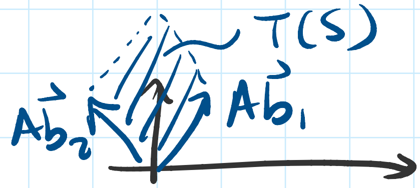
(n=2)

$$S = \{ s_1\vec{b}_1 + s_2\vec{b}_2 : 0 \le s_1 \le 1,\ 0 \le s_2 \le 1 \}$$

$T(S)$?

let $\vec{x} \in S$, $\vec{x} = s_1\vec{b}_1 + s_2\vec{b}_2$ $\left(\begin{matrix} 0 \le s_1 \le 1 \\ 0 \le s_2 \le 1 \end{matrix}\right)$

$$T(\vec{x}) = A(s_1\vec{b}_1 + s_2\vec{b}_2)$$
$$= s_1 A\vec{b}_1 + s_2 A\vec{b}_2$$



$$\Rightarrow T(S) = \{ s_1 A\vec{b}_1 + s_2 A\vec{b}_2 : 0 \le s_1 \le 1,\ 0 \le s_2 \le 1 \}$$

$$\text{Area}(T(S)) = |\det([A\vec{b}_1\ A\vec{b}_2])|$$
$$= |\det(A[\vec{b}_1\ \vec{b}_2])|$$
$$= |\det(A)\det([\vec{b}_1\ \vec{b}_2])|$$
$$= |\det(A)| \cdot |\det([\vec{b}_1\ \vec{b}_2])|$$
$$= |\det(A)| \cdot \text{Area}(S).$$

$\square$

More generally:

Thm:
- Let D be a region in $\mathbb{R}^2$ w/ finite area $\text{Area}(D) < \infty$.
- Then, the area of the image of D under a lin. transf. $T: \mathbb{R}^2 \to \mathbb{R}^2$ is given by

D under a lin. transf. $T: \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$\text{Area}(T(D)) = |\det(A)| \cdot \text{Area}(D)$$

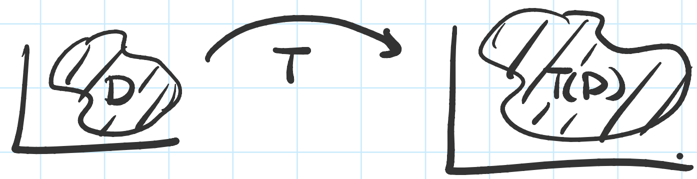
where A is the matrix assoc. to T

- Let V be a region in $\mathbb{R}^3$ w/ finite volume $\text{vol}(V) < \infty$.
- Then, the vol of the image of V under a lin. transf. $T: \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\text{Vol}(T(V)) = |\det(A)| \cdot \text{Vol}(V)$$

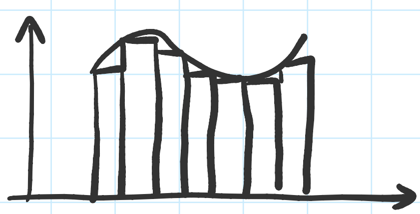
where A is the matrix assoc. to T

<u>pf sketch</u>:



calculus:

area under curves



$\square$

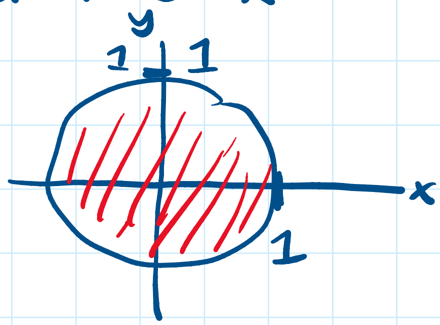
ex A Formula for the Area of an Ellipse

- unit disk D

y

1 1

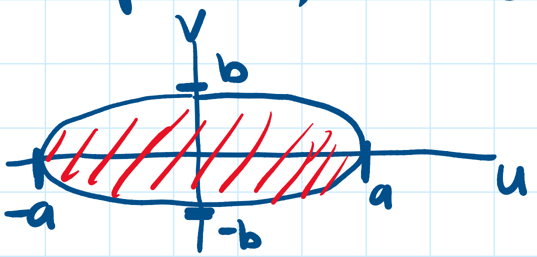
· unit disk D



$$D = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \le 1 \right\}$$

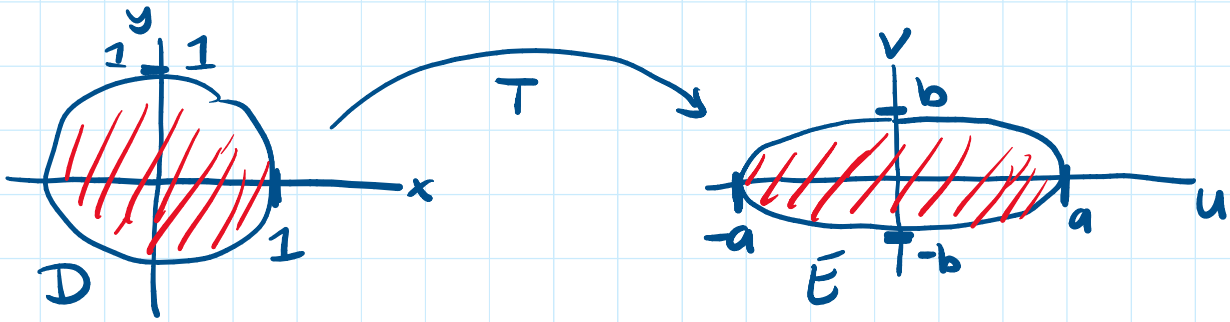
$$\text{Area}(D) = \pi.$$

· Ellipse w/ semi-axes $a, b > 0$



$$E = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} : \left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 \le 1 \right\}$$

$$\text{Area}(E)\ ?$$



Can I find a lin. transf. T s.t.

$$T(D) = E\ ?$$

stretch x axis by a $\rightarrow$ $u = ax$

stretch y axis by b $\rightarrow$ $v = by$

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \qquad A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$$

let T be the lin. transf. to A.

Prove $T(D) = E$.

Show $T(D) \subseteq E$ and $E \subseteq T(D)$

Prove $T(D) = E$.

Show $T(D) \subseteq E$ and $E \subseteq T(D)$

$(T(D) \subseteq E)$

let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in D$. Then $x^2 + y^2 \leq 1$.

$$T(\vec{x}) = \begin{bmatrix} ax \\ by \end{bmatrix} =: \begin{bmatrix} T(\vec{x})_1 \\ T(\vec{x})_2 \end{bmatrix}$$

$$\left(\frac{T(\vec{x})_1}{a}\right)^2 + \left(\frac{T(\vec{x})_2}{b}\right)^2$$

$$= \left(\frac{ax}{a}\right)^2 + \left(\frac{by}{b}\right)^2 = x^2 + y^2 \leq 1$$

$\Rightarrow T(\vec{x}) \in E$ for any $\vec{x} \in D$

$\Rightarrow T(D) \subseteq E$.

$(E \subseteq T(D))$

Since T is invertible, this statement is equivalent to showing $T^{-1}(E) \subseteq D$

let $\vec{u} = \begin{bmatrix} u \\ v \end{bmatrix} \in E$, $\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 \leq 1$

$$T^{-1}(\vec{u}) = A^{-1}\vec{u} = \begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u/a \\ v/b \end{bmatrix} = \begin{bmatrix} T^{-1}(\vec{u})_1 \\ T^{-1}(\vec{u})_2 \end{bmatrix}$$

$$(T^{-1}(\vec{u})_1)^2 + (T^{-1}(\vec{u})_2)^2$$

$$= \left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 \leq 1$$

$$= \left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 \le 1$$

$$\Rightarrow T^{-1}(\vec{u}) \in D$$

$$\Rightarrow T^{-1}(E) \subseteq D \iff E \subseteq T(D)$$

shown $T(D) \subseteq E$ & $E \subseteq T(D)$

$$\Rightarrow T(D) = E$$

$$\text{Area}(E) = \text{Area}(T(D))$$

prev. thm

$$= |\det(A)| \,\text{Area}(D)$$

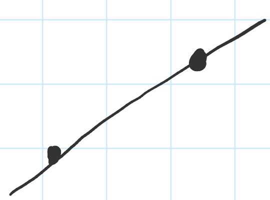
$$= \underbrace{\left|\det\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right)\right|}_{ab > 0} \underbrace{\text{Area}(D)}_{\pi}$$

$$= \pi ab$$

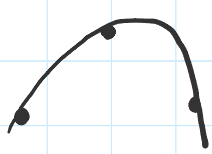
$\square$

Application of Linear Algebra:

Polynomial Interpolation



line passes through 2 pts



quadratic (parabola) passes through 3 pts

<u>Interpolation Problem:</u>

· Given n data points $\{(t_1, y_1), \ldots, (t_n, y_n)\}$,

- Given n data points $\{(t_1,y_1),\dots,(t_n,y_n)\}$, find a polynomial $p(t)$ passing through each data point, i.e., $p(t_j)=y_j \quad j=1,\dots,n$.
- Idea: use $\mathbb{P}_{n-1}$ to try to solve this because $\dim(\mathbb{P}_{n-1})=n=\#$ of data pts

ex (n=2) $\mathbb{P}_1=\{p(t)=c_0+c_1t : c_0,c_1\in\mathbb{R}\}$

data $(t_1,y_1),(t_2,y_2)$ $\quad p(t)=c_0+c_1t$

$$\begin{aligned} p(t_1)&=y_1 \\ p(t_2)&=y_2 \end{aligned} \Rightarrow \begin{aligned} c_0+c_1t_1&=y_1 \\ c_0+c_1t_2&=y_2 \end{aligned}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \end{bmatrix}}_{\substack{2\times2 \\ \text{Vandermonde} \\ \text{matrix } V}} \underbrace{\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}}_{\vec{c}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\vec{y}}$$

$$V\vec{c}=\vec{y}$$

When is V invertible?

$\det(V)=t_2-t_1\neq0$ as long as $t_1\neq t_2$, i.e. data pts measured at diff. times

$\Rightarrow \vec{c}=V^{-1}\vec{y}$

ex (n=3) $\mathbb{P}_2=\{c_0+c_1t+c_2t^2 : c_0,c_1,c_2\in\mathbb{R}\}$

ex/ $(n=3)$ $\mathbb{P}_2 = \{c_0 + c_1 t + c_2 t^2 : c_0, c_1, c_2 \in \mathbb{R}\}$

(t_1, y_1), (t_2, y_2), (t_3, y_3)

$p(t) = c_0 + c_1 t + c_2 t^2$ s.t.

$$\begin{aligned} p(t_1) &= y_1 \\ p(t_2) &= y_2 \\ p(t_3) &= y_3 \end{aligned} \implies \underbrace{\begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}}_{\vec{c}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\vec{y}}$$

$$\det(V) = (t_2 t_3^2 - t_3 t_2^2) - (t_1 t_3^2 - t_3 t_1^2) + (t_1 t_2^2 - t_2 t_1^2)$$

$$= (t_3 - t_2)(t_3 - t_1)(t_2 - t_1)$$

$\det(V) \neq 0$ as long as all t_1, t_2, t_3 are distinct. If so, $\vec{c} = V^{-1}\vec{y}$.

More generally, with n data pts

$(t_1, y_1), \ldots, (t_n, y_n)$

$\mathbb{P}_{n-1}$ $p(t) = c_0 + c_1 t + \ldots + c_{n-1} t^{n-1}$

$$\begin{aligned} p(t_j) &= y_j \\ j &= 1, \ldots, n \end{aligned} \implies \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & & & & \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix}$$

$j=1,\dots,n$

$$\underbrace{\begin{bmatrix} 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & & & & \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix}}_{n\times n} \begin{bmatrix} c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Vandermonde matrix

V

product (vs Σ)

$$\det(V) = \prod_{1\le i<j\le n} (t_j - t_i)$$

contains precisely one factor of $t_j - t_i$ for $i \neq j$

$\Rightarrow V$ invertible precisely when all the times $\{t_i\}_{i=1}^n$ for the data are distinct.

$\Rightarrow \vec{c} = V^{-1}\vec{y}$

$\square$

Eigenvalues & Eigenvector (ch 5)

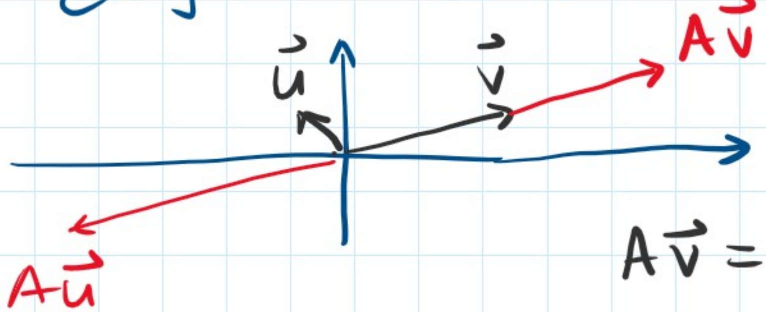
Consider diagonal matrices

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} \qquad A\vec{e}_j = a_{jj}\vec{e}_j$$

ex/ $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$A\vec{u} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$ $A\vec{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

$A\vec{w} = \begin{bmatrix} 12 \\ 6 \end{bmatrix} = 2\vec{w}$



$A\vec{v} = 2\vec{v}$

Def: An eigenvector of an $n \times n$ matrix A is a nonzero vector $\vec{x}$ s.t. $A\vec{x} = \lambda\vec{x}$ for some scalar λ.
If such an eigenvector $\vec{x} \neq 0$ exists s.t. $A\vec{x} = \lambda\vec{x}$, we call λ an eigenvalue of A and call $\vec{x}$ an eigenvector associated to λ.

Consider the eigenvalue-eigenvector:
find a nonzero vector $\vec{x}$ & scalar λ s.t.

$$A\vec{x} = \lambda\vec{x}$$

$$\iff A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$\iff A\vec{x} - \lambda I\vec{x} = \vec{0}$$

$$\iff (A - \lambda I)\vec{x} = \vec{0}$$

$\vec{x}$ solves $A\vec{x} = \lambda\vec{x} \iff \vec{x} \in \mathrm{nul}(A - \lambda I)$.

$\vec{x}$ is an eigenvector if ↰ + also nonzero.

The space of eigenvectors $\vec{x}$ corresp. to an eigenvalue (once adding back $\vec{0}$ to the set) is a subspace of $\mathbb{R}^n$, which we call the eigenspace assoc. to λ

ex/ Find a basis for the eigenspace corresp. to eigenvalue $\lambda = 2$ for

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 1 & 4 & 3 \\ 2 & 4 & 8 \end{bmatrix}$$

$$A - \underset{\substack{\Uparrow \\ 2}}{\lambda} I = A - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 2 & 3 \\ 1 & 4-\lambda & 3 \\ 2 & 4 & 8-\lambda \end{bmatrix}$$

$$\overset{\lambda=2}{\downarrow} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2 & x_3 free

$$\text{nul}(A-2I) = \left\{ \begin{bmatrix} -2x_2-3x_3 \\ x_2 \\ x_3 \end{bmatrix} : x_2, x_3 \in \mathbb{R} \right\}$$

$$= \left\{ x_2 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{= \vec{v}_1} + x_3 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{= \vec{v}_2} : x_2, x_3 \in \mathbb{R} \right\}$$

$\{\vec{v}_1, \vec{v}_2\}$ is a basis for the $\lambda=2$ eigenspace.

What about the zero eigenvalue? $\lambda=0$

$(A-\lambda I)\vec{x} = \vec{0}$

$A\vec{x} = \vec{0}$ ← nontrivial solution $\vec{x} \neq \vec{0}$

Thm matrix has an eigenvalue of 0

$\Leftrightarrow$ $A\vec{x} = \vec{0}$ has a nontrivial soln

$\Leftrightarrow$ A is not invertible.

$\blacksquare$