Practice Midterm Solutions

Friday, July 21, 2023 7:54 AM

1) Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

Is $\vec{b}$ is in the span of $\{\vec{v}_1, \vec{v}_2\}$

Augmented system

$$[\vec{v}_1 \quad \vec{v}_2 \mid \vec{b}]$$

$$= \left[\begin{array}{cc|c} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 3 & 4 & 5 \end{array}\right]$$

$$\xrightarrow[③ - 3① \to ③]{② + ① \to ②} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 7 & -1 \end{array}\right]$$

$$\xrightarrow{③ - 7② \to ③} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -22 \end{array}\right] \implies 0 = -22 \text{ impossible}$$

linear system is not consistent

no solution to $x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{b}$

$\Rightarrow \vec{b}$ is not in span$\{\vec{v}_1, \vec{v}_2\}$. $\square$

2) $$A = \begin{bmatrix} 1 & -3 & 3 & 4 \\ 2 & -6 & 0 & 2 \\ 2 & -6 & 1 & 3 \end{bmatrix}$$

basis for nul(A)

col(A)

$\begin{bmatrix} 2 & -6 & 1 & 3 \end{bmatrix}$ col(A)

$$\xrightarrow[\textcircled{3} - 2\textcircled{1} \to \textcircled{3}]{\textcircled{2} - 2\textcircled{1} \to \textcircled{2}} \begin{bmatrix} 1 & -3 & 3 & 4 \\ 0 & 0 & -6 & -6 \\ 0 & 0 & -5 & -5 \end{bmatrix}$$

$$\xrightarrow[\textcircled{3} - \frac{5}{6}\textcircled{2} \to \textcircled{3}]{} \begin{bmatrix} 1 & -3 & 3 & 4 \\ 0 & 0 & -6 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ an echelon form

$$\xrightarrow{\textcircled{3} \cdot \left(\frac{-1}{6}\right) \to \textcircled{3}} \begin{bmatrix} 1 & -3 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\textcircled{1} - 3\textcircled{2} \to \textcircled{1}} B = \begin{bmatrix} 1 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ reduced row echelon form

Pivot cols are cols 1 & 3

$\Rightarrow \{\vec{a}_1, \vec{a}_3\}$ is a basis for col(A)

$$= \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix} \right\}$$

nul(A) : solve $A\vec{x} = \vec{0}$

equivalent to $B\vec{x} = \vec{0}$ where nul(A): solve $A\vec{x} = \vec{0}$

equivalent to $B\vec{x} = \vec{0}$ where B is any row reduced matrix arising from A

$$B\vec{x} = \vec{0}: \quad \left[\begin{array}{cccc|c} \multicolumn{4}{c|}{\overbrace{\hspace{4cm}}^{B}} & \\ 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

x_2 & x_4 free

$$\begin{aligned} x_1 - 3x_2 + 0x_3 + x_4 &= 0 \\ 0x_1 + 0x_2 + 1x_3 + 1x_4 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\Rightarrow \text{nul}(A) = \left\{ \begin{bmatrix} 3x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} : x_2, x_4 \in \mathbb{R} \right\}$$

$$= \left\{ x_2 \underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_1} + x_4 \underbrace{\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\vec{v}_2} : x_2, x_4 \in \mathbb{R} \right\}$$

Claim: $\{\vec{v}_1, \vec{v}_2\}$ is a basis for nul(A).

Clearly spans nul(A).

Clearly spans $\text{nul}(A)$

$\vec{v}_1$ & $\vec{v}_2$ lin. ind.

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0} \Rightarrow \boxed{\text{either} \quad c_1 = 0 = c_2}$$

or

$$\cancel{c_1 \neq 0 \quad \vec{v}_1 = -\frac{c_2}{c_1}\vec{v}_2}$$

or

$$\cancel{c_2 \neq 0 \quad \vec{v}_2 = -\frac{c_1}{c_2}\vec{v}_1}$$

$\Rightarrow$ lin. ind.

$\Rightarrow \{\vec{v}_1, \vec{v}_2\}$ is a basis for $\text{nul}(A)$

Verify rank-nullity:

$$\underbrace{\dim(\text{col}(A))}_{=2} + \underbrace{\dim(\text{nul}(A))}_{=2} = \#\text{ columns of } A$$

$2 + 2 = 4$ ✓

∎

3) $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{b}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\mathcal{B}$ is a basis for $\mathbb{R}^3 \Longleftrightarrow$

$B = [\vec{b}_1 \; \vec{b}_2 \; \vec{b}_3]$ is invertible.

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

$$\det(B) = 1(1-0) - 0(0-2) + (-1)(0-1)$$
$$= 1 + 1 = 2 \neq 0$$

$\Rightarrow B$ is invertible

$\Rightarrow$ By the inv. matrix theorem, the columns of B, i.e. $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ form a basis for $\mathbb{R}^3$.

$\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ find $[\vec{x}]_{\mathcal{B}}$

$$\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + c_3 \vec{b}_3$$
$$= \underbrace{[\vec{b}_1 \; \vec{b}_2 \; \vec{b}_3]}_{B} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{[\vec{x}]_{\mathcal{B}}}.$$

Solve

$[B \,\vdots\, \vec{x}]$

$= [\; 1 \quad 0 \quad \vdots \; 1 \quad 2 \;]$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{③ + ① \to ③} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 5 \end{array}\right]$$

$$\xrightarrow{③ - 2① \to ③} \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 \end{array}\right] \end{array}$$

3rd row $\quad 2c_3 = 5$

$$c_3 = 5/2$$

2nd row $\quad c_2 = 0$

1st row $\quad c_1 + c_3 = 2$

$$c_1 = 2 - c_3$$

$$= 2 - \frac{5}{2} = -\frac{1}{2}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 5/2 \end{bmatrix}$$

$\square$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ \cdots \end{array}\right] \xrightarrow{③ \cdot \frac{1}{2} \to ③} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ \cdots \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 \end{array}\right] \xrightarrow{r_3 \cdot \frac{1}{2} \to r_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5/2 \end{array}\right]$$

$$\xrightarrow{r_1 - r_3 \to r_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5/2 \end{array}\right]$$

4) $\mathbb{P}_4 = \{ p(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 : c_0, c_1, c_2, c_3, c_4 \in \mathbb{R} \}$

$V = \{ p(t) = a + bt^2 + ct^4 ; a, b, c \in \mathbb{R} \}$

<u>Subspaces</u>: U is a subspace of vector space W

(i) Zero vector $\vec{0}$ in W is in U, $\vec{0} \in U$

(ii) U is closed under vector addition

$$\vec{u}_1, \vec{u}_2 \in U \Rightarrow \vec{u}_1 + \vec{u}_2 \in U$$

(iii) U is closed under scalar mult.

$$c \in \mathbb{R}, \vec{u} \in U \Rightarrow c\vec{u} \in U$$

$p(t) = a + bt^2 + ct^2$

$a = b = c = 0 \Rightarrow$ zero polynomial is in V (i) ✓

let $p(t) = a + bt^2 + ct^4 \in V$
$q(t) = d + et^2 + ft^4 \in V$
be two arbitrary poly. in V

$p(t) + q(t)$
$= (a+d) + (b+e)t^2 + (c+f)t^4 \in V$
$\Rightarrow$ V is closed under vector add (ii) ✓

let $p(t) = a + bt^2 + ct^4 \in V$ and $k \in \mathbb{R}$

$kp(t) = ka + (kb)t^2 + (kc)t^4 \in V$
$\Rightarrow$ V is closed und scalar mult (iii) ✓
$\Rightarrow$ V subspace of W ✓

$\dim(V) = \#$ of elements in a basis for V

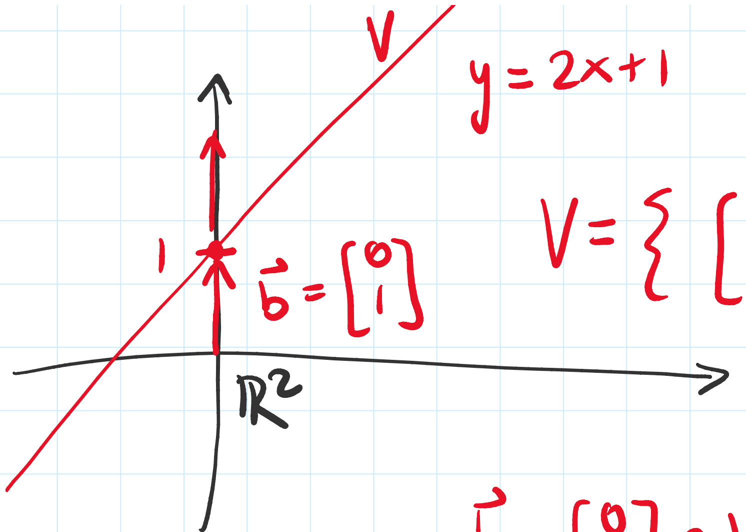
V has a basis $\{1, t^2, t^4\}$

$\dim(V) = 3$

$\square$

$\cancel{V}$ $y = 2x + 1$

$\square$



$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = 2x+1 \right\}$ is a subset of $\mathbb{R}^2$

$\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V$ $\quad$ $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V$

$\vec{b} + \vec{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \begin{matrix} = x \\ = y \end{matrix}$ but $y \neq 2x+1$

$\Rightarrow \vec{b} + \vec{b} \notin V$

$\Rightarrow V$ is not closed under vector addition

5) $T : V \longrightarrow W$ bijective (i.e. invertible) linear transf.

V has basis $\{\vec{v}_1, \dots, \vec{v}_n\} \Rightarrow \dim(V) = n$

Show $\{\underbrace{T(\vec{v}_1), \dots, T(\vec{v}_n)}\}$ is a basis for W

show linearly ind. & spans W.

(lin. ind.)

Suppose have a dependence relation

Suppose have a dependence relation

$$c_1 T(\vec{v}_1) + \dots + c_n T(\vec{v}_n) = \vec{0}$$

using linearity of T,

$$T(\overbrace{c_1 \vec{v}_1 + \dots + c_n \vec{v}_n}^{\in \ker(T)}) = \vec{0} \leftarrow \text{homog. eqn}$$

but, T is invertible, so $\ker(T) = \{\vec{0}\}$

$$\Rightarrow c_1 \vec{v}_1 + \dots c_n \vec{v}_n = \vec{0}$$

but $\{\vec{v}_1, \dots, \vec{v}_n\}$ form a basis for V
& so are linearly independent $\leftarrow$

$$\Rightarrow c_1 = 0, \dots, c_n = 0$$

$\Rightarrow \{T(\vec{v}_1), \dots, T(\vec{v}_n)\}$ is lin. ind.

(span)

let $\vec{y} \in W$. I have to show I can write $\vec{y}$ as a lin. comb of $T(\vec{v}_1), \dots, T(\vec{v}_n)$.

Since T is surjective, there exists $\vec{x} \in V$ s.t. $T(\vec{x}) = \vec{y}$

$\vec{x} \in V$ s.t. $T(\vec{x}) = \vec{y}$

Because $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for V,
there exists $c_1, \dots, c_n$ s.t.
$\vec{x} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$

$$T(c_1 \vec{v}_1 + \dots + c_n \vec{v}_n) = \vec{y}$$

linearity of T

$$c_1 T(\vec{v}_1) + \dots + c_n T(\vec{v}_n) = \vec{y}$$

$\Rightarrow \{T(\vec{v}_1), \dots, T(\vec{v}_n)\}$ spans W.

$\Rightarrow \{T(\vec{v}_1), \dots, T(\vec{v}_n)\}$ is a basis for W.

$\Rightarrow \dim(W) = n$

$\Rightarrow \dim(V) = \dim(W)$.

$\square$

Let $\mathbb{M}_{m \times n}$ be the set of all $m \times n$ matrices.

Show that $\mathbb{M}_{m \times n}$ (rows, cols) is a vector space where vector addition is matrix addition & scalar multiplication is usual where vector addition is matrix addition & scalar multiplication is usual scalar-matrix multiplication.

- $\text{Zero} = \overbrace{\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}}^{n} \Big\} m$

For any $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$

$A + \text{Zero} = A.$ ✓

- Closure under vector addition

$A, B \in M_{m \times n}$.

$$A + B = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{bmatrix} \in M_{m \times n}$$

M closed under vector addition.

Scalar mult: $c \in \mathbb{R}$, $A \in M_{m \times n}$

$cA = \begin{bmatrix} a_{11} & \cdots & a_{1n} \end{bmatrix} \quad (ca_{11} \cdots ca_{1n})$

scalar mult: $c \in \mathbb{R}$, $A \in M_{m\times n}$

$$cA = c\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{bmatrix}$$

$M_{m\times n}$

$\Rightarrow$ closed under scalar multiplication

$\Rightarrow$ $M_{m\times n}$ is a vector space.

$\dim(M_{m\times n}) = mn$

$M(i,j)$ = Ts the matrix with 1 in the (i,j) entry and zero everywhere else

$i = 1, \dots, m$

$j = 1, \dots, n$.

ex/ $m=3$ $n=2$ $\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$ $M_{3\times 2}$

$M(1,1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $M(1,2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$M(2,1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$ $M(2,2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$M(3,1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$ $M(3,2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

$3\cdot 2 = 6$

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