Math 18 Summer Session 1 2023: Homework 2

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Due Wednesday, July 19, 11:59 pm.

Remark. Problems written as "Exercise X.Y.Z" are from the textbook, section X.Y exercise Z. For example, Exercise 1.2.4 denotes exercise 4 of section 1.2. For problems referring to a figure, find the question in the textbook for the corresponding figure. Make sure to show all of your work and steps; credit will not be given for just stating an answer.

Problem 1 Linear Transformations from $\mathbb{R}$ to $\mathbb{R}$

The goal of this problem is to introduce you to proofs by taking your knowledge of linear transformations (that we learned in the arbitrary $m \times n$ setting) and seeing it in the context of the simplest case, maps from 1-dimensional $\mathbb{R}$ to itself.

Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = ax + b,$$

for some given $a, b \in \mathbb{R}$.

(a)

Prove that f is a bijection if and only if $a \neq 0$ (whereas $b \in \mathbb{R}$ can be arbitrary).

Hint: First, although pictures are not proofs, it helps guide your intuition for a proof. Whenever you can draw a picture in mathematics, try to do so: How does the graph of f look like when $a = 0$ versus when $a \neq 0$?

Now, onto the proof. You must prove both directions:

For one direction, assume that f is a bijection. You want to then prove that $a \neq 0$. To do this, assume for the sake of contradiction that $a = 0$. Then, prove there is no possible way for f to be a surjection and hence, not a bijection. Thus, by contradiction, a cannot be 0.

For the other direction, assume that $a \neq 0$. Then, prove directly that $f : \mathbb{R} \to \mathbb{R}$ is both injective and surjective (i.e., for injectivity, show that different inputs get mapped to different outputs, and for injectivity, show that all possible values in the codomains are reached).

(b)

Prove that f is a linear transformation if and only if $b = 0$ (whereas $a \in \mathbb{R}$ can be arbitrary).

Hint: This part follows from a direct computation:

For one direction, show that if $b = 0$, then

$$f(c_1 u + c_2 v) = c_1 f(u) + c_2 f(v) \text{ for all } c_1, c_2, u, v \in \mathbb{R};$$

i.e., f is a linear transformation.

For the other direction, show that if $b \neq 0$, then

$$f(c_1 u + c_2 v) \neq c_1 f(u) + c_2 f(v)$$

for some choices of c_1, c_2, u, v (pick a choice); i.e., f is not a linear transformation.

(c)

Assuming $b = 0$, i.e., f is a linear transformation, what is the matrix associated to f? Combine sections (a) and (b) to give a precise characterization of the set of invertible linear transformations $f : \mathbb{R} \to \mathbb{R}$ in terms of the matrix associated to f.

Hint: What does a 1×1 matrix look like?

Problem 2 Inner Products

We can view vectors in $\mathbb{R}^n$ as $n \times 1$ matrices . Thus, the transpose of a vector $\vec{v} \in \mathbb{R}^n$ can be viewed as an $1 \times n$ matrix. For two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, the product $\vec{u}^T \vec{v}$ is called an inner product on $\mathbb{R}^n$; it produces a 1×1 matrix which is usually just written as a real number without the brackets (we will see these inner products later in the course).

(a) Let

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

Compute the inner product $\vec{u}^T \vec{v}$.

(b) Let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^n$. How are $\vec{x}^T \vec{y}$ and $\vec{y}^T \vec{x}$ related?

Hint: Write out a formula for $\vec{x}^T \vec{y}$ and for $\vec{y}^T \vec{x}$ in terms of the components of $\vec{x}, \vec{y}$ and compare these formulas.

Problem 3 Exercise 2.2.2

Compute the determinant of the matrix to determine if it is invertible. If it is, find the inverse.

$$A = \begin{bmatrix} 3 & 1 \\ 7 & 2 \end{bmatrix}.$$

Problem 4 Exercise 2.2.41

Compute the determinant of A to determine if it is invertible. If it is, use the algorithm discussed in class to find the inverse of this matrix. Recall the algorithm is to row reduce

$$\begin{bmatrix} A & I \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} I & A^{-1} \end{bmatrix},$$

where

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}.$$

Problem 5 Exercise 2.3.32

If L is an $n \times n$ matrix and $L\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$, do the columns of L span $\mathbb{R}^n$? Why?

Hint: How many pivots must L have?

Problem 6 Exercise 4.1.8

Consider the subset of the vector space of polynomials of degree n, $\mathbb{P}_n$, defined by

$$H = \{p(x) \in \mathbb{P}_n : p(0) = 0\},$$

i.e., H consists of all polynomials of degree n that evaluate to 0 at $x = 0$. Show that H is a subspace of $\mathbb{P}_n$, by verifying that it closed under vector addition and scalar multiplication.

Problem 7 Exercise 4.1.10

Let

$$H = \left\{ \begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Show that H is a subspace of $\mathbb{R}^3$, by finding a vector $\vec{v} \in \mathbb{R}^3$ such that $H = \operatorname{span}\{\vec{v}\}$.

Problem 8 Exercise 4.2.4

Find a set of vectors which span the null space of A,

$$A = \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

Problem 9 Practice with column and null spaces

Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 1 \\ 3 & -1 & 3 \end{bmatrix}.$$

Is $\operatorname{col}(A)$ all of $\mathbb{R}^3$? Why or why not?
Is $\operatorname{nul}(A)$ all of $\mathbb{R}^3$? Why or why not?
Hint: Row reduce A. What do the number of pivots tell you about the dimension of these spaces?

Problem 10 Exercise 4.3.14

Assume that A is row equivalent to B. Find bases for $\operatorname{nul}(A), \operatorname{col}(A), \operatorname{row}(A)$.

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Problem 11 Invertible Linear Transformations Preserve Bases

Let $\{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Prove that the collection of vectors

$$\{T(\vec{b}_1), \dots, T(\vec{b}_n)\}$$

is also a basis for $\mathbb{R}^n$; i.e., applying an invertible linear transformation to a basis is still a basis.

Hint: Let A denote the $n \times n$ matrix associated to T; note that A is an invertible matrix since T is invertible. Using A, the above collection of vectors can be expressed

$$\{A\vec{b}_1, \dots, A\vec{b}_n\}.$$

Now, we want to show that this collection of vectors spans $\mathbb{R}^n$ and is linearly independent. To do this, consider the matrix whose columns are given by these vectors:

$$[A\vec{b}_1 \ \cdots \ A\vec{b}_n].$$

Using matrix multiplication, this can be expressed as a product

$$[A\vec{b}_1 \ \cdots \ A\vec{b}_n] = AB,$$

where A is the matrix associated to T and B is the matrix whose columns are $\vec{b}_1, \ldots, \vec{b}_n$. From class, we know that AB is invertible, since A and B are. What does this tell you about the column space of AB and the null space of AB? Use this to argue that the columns of AB span $\mathbb{R}^n$ and are linearly independent, and hence, form a basis for $\mathbb{R}^n$.

Problem 12 Exercise 4.4.8

Determine the coordinate vector $[\vec{x}]_B$ of $\vec{x}$ relative to the basis $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$.

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \vec{x} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}.$$

Problem 13 Exercise 4.5.4

Find a basis for the subspace and state its dimension.

$$\left\{ \begin{bmatrix} a+b \\ 2a \\ 3a-b \\ -b \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Problem 14 Exercise 4.5.36

Suppose a 5×6 matrix A has four pivot columns. What is nullity A? Is $\mathrm{col}(A) = \mathbb{R}^4$? Why or why not?