Math 18 Summer Session 1 2023: Homework 4

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Due Wednesday, August 2, 11:59 pm.

Remark. Problems written as "Exercise X.Y.Z" are from the textbook, section X.Y exercise Z. For example, Exercise 1.2.4 denotes exercise 4 of section 1.2. For problems referring to a figure, find the question in the textbook for the corresponding figure. Make sure to show all of your work and steps; credit will not be given for just stating an answer.

Problem 1 Exercise 5.3.14

Determine whether the matrix A is diagonalizable. If it is, diagonalize it, i.e., find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

$$A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

Problem 2 Polynomial Functions of Similar Matrices

The usual way we think of polynomials is as taking in an input of a number x, taking nonnegative integer powers of that number, multiplying those powers by some coefficients, and adding them together, i.e.,

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n.$$

Note the 0^{th} power of a number x is just $x^0 = 1$, which corresponds to the constant term above in $p(x)$.

We are going to extend this idea to matrices. We know how to add matrices of the same dimension. We also know how to multiply matrices. In particular, for two $n \times n$ matrices, we know their product is an $n \times n$ matrix. Thus, if A is an $n \times n$ matrix, we can define its k^{th} power, where k is a nonnegative integer, as multiplying it by itself k times:

$$A^k = \underbrace{A \cdots A}_{k \text{ times}},$$

where we define $A^0 = I$ (a square matrix raised to the 0^{th} power is the identity matrix, in analogy to numbers raised to the 0^{th} power being 1), and for example, $A^1 = A$, $A^2 = AA$, $A^3 = AAA$, etc.

Since we know how to add and take powers of square matrices, this means we can define polynomials as functions of square matrices, which take in a square matrix and returns a square matrix. A polynomial function of a square matrix A has the form

$$p(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_n A^n.$$

Let A and B be similar, i.e., there exists an invertible P such that $B = PAP^{-1}$. Let $p(A)$ be a polynomial function on square matrices. Show that

$$p(B) = Pp(A)P^{-1},$$

i.e., that $p(B)$ and $p(A)$ are also similar, with the same similarity relation as B and A.

Additionally, use this to evaluate the polynomial function $p(A) = A^8$, where

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Hint. For the first part of the problem, by linearity, it suffices to prove it for the monomials, i.e., that the k^{th} monomial $p_k(A) = A^k$ satisfies $p_k(B) = Pp_k(A)P^{-1}$ for every nonnegative integer k. You can show this by induction. Show that it holds for the base case $k = 0$, where $p_0(A) = I$ (i.e., the monomial p_0 of degree 0 outputs I regardless of its input). Then, assume that $p_k(B) = Pp_k(A)P^{-1}$ holds for a nonnegative integer k. Show that it holds for the next nonnegative integer $k+1$, $p_{k+1}(B) = Pp_{k+1}(A)P^{-1}$.

For the second part of the problem, diagonalize the matrix A so that it has the form $A = PDP^{-1}$. Then, use the previous part that you proved, noting that for a diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix},$$

its k^{th} power is just given by raising each of the entries to that power:

$$D^k = \begin{bmatrix} d_1^k & 0 \\ 0 & d_2^k \end{bmatrix}.$$

Problem 3 Practice with Finding Eigenvalues and Eigenvectors of Linear Transformations

Let $\mathcal{B} = \{1, t, t^2\}$ be the monomial basis of $\mathbb{P}_2$. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation defined by

$$T(c_0 + c_1 t + c_2 t^2) = (c_0 - 2c_1 + c_2) + (c_0 + 2c_1 + c_2)t^2.$$

Find the eigenvalues of T and one eigenvector corresponding to each eigenvalue. That is, for each eigenvalue λ of T, find a nonzero polynomial of degree at most two, $p(t) \in \mathbb{P}_2$, such that

$$T(p(t)) = \lambda p(t).$$

Hint. This is similar to the example we did in lecture. Instead of working on the abstract vector space $\mathbb{P}_2$, find the matrix representation $[T]_{\mathcal{B}}$ of T relative to the monomial basis $\mathcal{B}$. Then, find the eigenvalues and eigenvectors of the matrix $[T]_{\mathcal{B}}$, which as discussed in lecture, tells you the eigenvalues and eigenvectors of T.

Problem 4 Exercise 6.1.10

Find a unit vector in the direction of

$$\begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}.$$

Problem 5 Exercise 6.1.14

Find the distance between the vectors

$$\vec{u} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} -4 \\ -1 \\ 4 \end{bmatrix}.$$

Problem 6 The Orthogonal Complement is a Subspace

Let W be a subspace of $\mathbb{R}^n$. Show that the orthogonal complement $W^\perp$, i.e., the set of all vectors orthogonal to W, is a subspace of $\mathbb{R}^n$.

Hint. As usual, you need to check 3 properties: the zero vector is in $W^\perp$, $W^\perp$ is closed under vector addition (if you take two vectors orthogonal to every vector in W, show that their sum is also orthogonal to every vector in W), and $W^\perp$ is closed under scalar multiplication (if you take a vector orthogonal to every vector in W, show that if you multiply that vector by a scalar, it is still orthogonal to every vector in W).

Problem 7 Exercise 6.2.2

Determine if the set of vectors is orthogonal.

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}.$$

Problem 8 Exercise 6.2.8

Show that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis for $\mathbb{R}^2$. Then, express $\vec{x}$ as a linear combination of $\{\vec{u}_1, \vec{u}_2\}$.

$$\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}, \text{ and } \vec{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}.$$

Problem 9 Exercise 6.3.4

Verify that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set and then find the orthogonal projection of $\vec{y}$ onto $\text{span}\{\vec{u}_1, \vec{u}_2\}$.

$$\vec{u}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}, \vec{y} = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}.$$

Problem 10 Exercise 6.3.8

Let W be the subspace spanned by $\vec{u}_1$ and $\vec{u}_2$. Write $\vec{y}$ as the sum of a vector in W and a vector orthogonal to W, i.e., in the form $\vec{y} = \vec{y}_{||} + \vec{y}_\perp$ where $\vec{y}_{||} \in W, \vec{y}_\perp \in W^\perp$.

$$\vec{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}.$$

Problem 11 The Sum of a Subspace and its Orthogonal Complement

Let U and V be subspaces of $\mathbb{R}^n$. We define the sum of these two subspaces, $U + V$, as the set of all sums of vectors in U with vectors in V, i.e.,

$$U + V = \{\vec{u} + \vec{v} : \vec{u} \in U, \vec{v} \in V\}.$$

Show that $U + V$ is a subspace of $\mathbb{R}^n$.

Now, let W be a subspace of $\mathbb{R}^n$ and let $W^\perp$ be its orthogonal complement. Explain why $W + W^\perp = \mathbb{R}^n$. Furthermore, show that $\dim(W) + \dim(W^\perp) = \dim(\mathbb{R}^n)$.

Hint. To show $U + V$ is a subspace, check the usual three properties required.

To see why $W + W^\perp = \mathbb{R}^n$, use the orthogonal decomposition theorem.

To show that $\dim(W) + \dim(W^\perp) = \dim(\mathbb{R}^n)$, let $\{\vec{x}_1, \dots, \vec{x}_p\}$ be an orthogonal basis for W and let $\{\vec{y}_1, \dots, \vec{y}_r\}$ be an orthogonal basis for $W^\perp$. Consider the set of vectors given by combining these bases together:

$$\mathcal{B} = \{\vec{x}_1, \dots, \vec{x}_p, \vec{y}_1, \dots, \vec{y}_r\}.$$

Show that $\mathcal{B}$ is an orthogonal set (i.e., all of the vectors in $\mathcal{B}$ are orthogonal to each other). Use this to argue that $\mathcal{B}$ is a linearly independent set of vectors. Then, explain why $\mathcal{B}$ spans $\mathbb{R}^n$ (use the orthogonal decomposition theorem). This shows that $\mathcal{B}$ is a basis for $\mathbb{R}^n$. Use this to conclude that $\dim(W) + \dim(W^\perp) = \dim(\mathbb{R}^n)$.

Problem 12 Practice with Orthogonal Matrices

Consider the 4×4 matrix

$$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Verify that U is an orthogonal matrix, i.e., that the columns of U are orthonormal (i.e., check that the columns $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$ of U satisfy $\vec{u}_i \cdot \vec{u}_j$ equals 0 if $i \neq j$ and equals 1 if $i = j$).

Then, since U is orthogonal, we know that its inverse is equal to its transpose, $U^{-1} = U^T$. Use this to solve the linear system $U\vec{x} = \vec{y}$ where

$$\vec{y} = \begin{bmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \\ 1 \end{bmatrix}.$$

Problem 13 Exercise 6.4.4

Let $\{\vec{v}_1, \vec{v}_2\}$ be a basis for a subspace W, where

$$\vec{v}_1 = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix}.$$

Use the Gram–Schmidt process to produce an orthogonal basis for W.

Problem 14 More practice with the Gram–Schmidt process

Consider the set of vectors in $\mathbb{R}^3$ given by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.$$

Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

Then, use the Gram–Schmidt process to find an orthogonal basis of $\mathbb{R}^3$, $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$, with the property

$$\begin{aligned} \text{span}\{\vec{v}_1\} &= \text{span}\{\vec{u}_1\} \\ \text{span}\{\vec{v}_1, \vec{v}_2\} &= \text{span}\{\vec{u}_1, \vec{u}_2\}, \\ \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} &= \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}. \end{aligned}$$