Math 20D Summer Session 1 2022: Homework 3

Instructor: Brian Tran

Due Wednesday, July 20, 11:59 pm.

Remark. Problems written as "Exercise X.Y.Z" are from the textbook, section X.Y exercise Z. For example, Exercise 1.2.4 denotes exercise 4 of section 1.2. For problems referring to a figure or result, find the question in the textbook for the corresponding figure or result. Make sure to show all of your work and steps; credit will not be given for just stating an answer.

Problem 1 Converting an IVP at time t_0 to an IVP at time 0

Recall that we used Laplace transforms to solve initial value problems given at time $t = 0$. While one might initially think that this is not sufficient to solve initial value problems given at some other time $t = t_0$, it turns out that any IVP at time t_0 can be transformed into an initial value problem given at time 0. We will explore this in this problem.

From the first homework, we know that any n^{th}-order DE can be transformed to a system of first-order DEs, so it suffices to prove this fact for first-order DEs. Consider the general form for a first-order IVP

$$\begin{aligned} \frac{d}{dt}x(t) &= f(t, x(t)), \\ x(t_0) &= x_0. \end{aligned}$$

We want to transform this to an IVP at time 0, so define a new time variable $\tau = t - t_0$, which translates the original time t by the amount t_0. Observe that this transformation is invertible, $t = \tau + t_0$, so it is equivalent to work with this new time variable. Define a new dependent variable $y(\tau) = x(\tau + t_0) = x(t)$. **Show that with these new variables, the original IVP at time $t = t_0$ can be transformed to an IVP at time $\tau = 0$ of the form**

$$\begin{aligned} \frac{d}{d\tau}y(\tau) &= \dots, \\ y(0) &= \dots, \end{aligned}$$

and fill in the right hand sides "…". Hint: to compute the derivative $dy/d\tau$, use the chain rule and the fact that $y(\tau) = x(t)$.

Problem 2 Exercise 7.2.12

Use Definition 1 to determine the Laplace transform of the given function (this includes stating the domain where the Laplace transform is defined)

$$f(t) = \begin{cases} e^{2t}, & 0 < t < 3, \\ 1, & t > 3. \end{cases}$$

Problem 3 Exercise 7.2.16

Use the Laplace transform table and the linearity of the Laplace transform to compute the given Laplace transform (this includes stating the domain where the Laplace transform is defined)

$$\mathcal{L}\Big(t^2 - 3t - 2e^{-t}\sin(3t)\Big)(s).$$

Problem 4 Exercise 7.3.6

Determine the Laplace transform of the given function using the Laplace transform table and the properties of Laplace transforms (this includes stating the domain where the Laplace transform is defined)

$$e^{-2t}\sin(2t) + e^{3t}t^2.$$

Problem 5 Exercise 7.4.4

Determine the inverse Laplace transform of the given function

$$\frac{4}{s^2+9}.$$

Problem 6 Exercise 7.4.10

Determine the inverse Laplace transform of the given function

$$\frac{s-1}{2s^2+s+6}.$$

Remark. For all of the following IVPs, you must solve them using the method of Laplace transforms to receive credit, even if there is another method to solve to the problem.

Problem 7 Exercise 7.5.4

Solve the given IVP using the method of Laplace transforms.

$$\begin{aligned} y'' + 6y' + 5y &= 12e^t \\ y(0) &= -1, \\ y'(0) &= 7. \end{aligned}$$

Problem 8 Exercise 7.5.8

Solve the given IVP using the method of Laplace transforms.

$$\begin{aligned} y'' + 4y &= 4t^2 - 4t + 10 \\ y(0) &= 0, \\ y'(0) &= 3. \end{aligned}$$

Problem 9 Exercise 7.5.25

Solve the given third-order IVP for $y(t)$ using the method of Laplace transforms.

$$\begin{aligned} y''' - y'' + y' &= 0, \\ y(0) &= 1, \\ y'(0) &= 1, \\ y''(0) &= 3. \end{aligned}$$

Problem 10 Exercise 7.6.18

Determine the inverse Laplace transform of the given function.

$$\frac{e^{-s}(3s^2 - s + 3)}{(s-1)(s^2+1)}.$$

Problem 11 Exercise 7.6.36

The unit triangular pulse $\Lambda(t)$ is defined by

$$\Lambda(t) = \begin{cases} 0, & t < 0, \\ 2t, & 0 < t < 1/2, \\ 2 - 2t, & 1/2 < t < 1, \\ 0, & t > 1. \end{cases}$$

(a) Sketch the graph of $\Lambda(t)$. Why is it so named?
(b) Show that $\Lambda(t) = \int_{-\infty}^{t} 2\Big(\Pi_{0,1/2}(\tau) - \Pi_{1/2,1}(\tau)\Big)d\tau$.
(c) Find the Laplace transform of $\Lambda(t)$.

Problem 12 Solving an IVP with a discontinuous inhomogeneity

Using the method of Laplace transforms, solve the following IVP:

$$\begin{aligned} \frac{dx}{dt} &= g(t), \\ x(0) &= 1, \end{aligned}$$

where

$$g(t) = \begin{cases} 1, & 0 < t < 1, \\ t, & 1 < t < 2, \\ e^t, & t > 2. \end{cases}$$

Graph your solution on the interval $[0, 3]$. You should see that it is continuous.