Math 20D Summer Session 1 2022: Homework 4

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Due Thursday, July 28, 11:59 pm.

Remark. Problems written as "Exercise X.Y.Z" are from the textbook, section X.Y exercise Z. For example, Exercise 1.2.4 denotes exercise 4 of section 1.2. For problems referring to a figure or result, find the question in the textbook for the corresponding figure or result. Make sure to show all of your work and steps; credit will not be given for just stating an answer.

Problem 1 Exercise 7.8.14

Find the Laplace transform of

$$f(t) = \int_0^t e^v \sin(t - v) dv.$$

Problem 2 Exercise 7.8.24

Find the transfer function $H(s)$ for the system (in lecture, I used the notation $\tilde{h}(s)$) and the impulse response function $h(t)$, and give a formula for the solution to the IVP (i.e., the formula should work for any inhomogeneity $g(t)$)

$$y'' - 9y = g(t),$$
$$y(0) = 2,$$
$$y'(0) = 0.$$

Problem 3 Asymptotic Stability

A linear system is said to be asymptotically stable if any solution $x(t)$ of the linear system decays to zero at infinity; i.e., $\lim_{t\to\infty} x(t) = 0$. For example, in class, we saw that the damped harmonic oscillator is asymptotically stable, because as time increases, friction dissipates energy in the system until the system approaches equilibrium. In essence, this is because adding a friction term to the oscillator differential equation introduced exponential decay to the oscillator.

Let g be a continuous function of exponential order 0. **Show that the linear system**

$$ax'' + bx' + cx = g(t),$$

is asymptotically stable if the real parts of the roots of the characteristic polynomial $ar^2 + br + c$ **are strictly less than zero.** You can for simplicity assume that the roots are complex (i.e., $b^2 - 4ac < 0$); this assumption is not necessary to prove the result, but it is just so you don't have to consider the three cases (real distinct roots, repeated roots, and complex roots) since the proof for all of the cases are similar.

Hint: Solve for the impulse response function $h(t)$. Once we have this, we can express any solution to the above system as

$$x(t) = C_1 x_1(t) + C_2 x_2(t) + \int_0^t h(t - \tau) g(\tau) d\tau,$$

where $C_1x_1(t) + C_2x_2(t)$ is the general solution to the homogeneous equation. Show that when the real parts of the roots of the characteristic equation are strictly less than zero, $x_1(t)$ and $x_2(t)$ go to zero as $t \to \infty$ (because they are exponentially decaying). Similarly, show that the limit

$$\lim_{t\to\infty} \int_0^t h(t-\tau)g(\tau)d\tau$$

is zero when the real parts of the roots are strictly less than zero (by a similar exponential decay argument).

Problem 4 Exercise 7.9.14

Solve the given distributional initial value problem.

$$\begin{aligned} y'' + 2y' + 2y &= \delta(t-\pi), \\ y(0) &= 1, \\ y'(0) &= 1. \end{aligned}$$

Problem 5 Exercise 8.3.26

Find at least the first four nonzero terms in a power series expansion about $x = 0$ for the solution to the given IVP

$$\begin{aligned} (x^2 - x + 1)y'' - y' - y &= 0, \\ y(0) &= 0, \\ y'(0) &= 1. \end{aligned}$$

Problem 6 Exercise 8.3.28

Find at least the first four nonzero terms in a power series expansion about $x = 0$ for the solution to the given IVP

$$\begin{aligned} y'' + (x-2)y' - y &= 0, \\ y(0) &= -1, \\ y'(0) &= 0. \end{aligned}$$

Problem 7 Exercise 9.5.12

Find a general solution of the system $\vec{x}\,'(t) = A\vec{x}(t)$ for the given matrix A.

$$A = \begin{pmatrix} 1 & 3 \\ 12 & 1 \end{pmatrix}.$$

Problem 8 Exercise 9.5.31

Solve the given IVP.

$$\begin{aligned} \vec{x}\,'(t) &= \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \vec{x}(t), \\ \vec{x}(0) &= \begin{pmatrix} 3 \\ 1 \end{pmatrix}. \end{aligned}$$

Problem 9 Exercise 9.6.13(a)

Solve the given IVP.

$$\vec{x}\,'(t) = \begin{pmatrix} -3 & -1 \\ 2 & -1 \end{pmatrix} \vec{x}(t),$$
$$\vec{x}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Problem 10 Converting an n^{th}-order linear DE to a linear system of DEs of dimension n

In Homework 1, we saw that any n^{th}-order differential equation can be converted to a system of n first-order differential equations. A special case of this is that any n^{th}-order linear differential equation can be converted to a first-order linear system of differential equations of dimension n. We will not prove this fact in full generality (although the proof is analogous to what we do in this problem), but we will just consider the $n = 3$ case, for concreteness.

Consider the general form for a third-order linear differential equation

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = f(t).$$

Define the new variables $v(t) = x'(t)$ and $a(t) = x''(t)$. Define the vector-valued function

$$\vec{y}(t) = \begin{pmatrix} x(t) \\ v(t) \\ a(t) \end{pmatrix}.$$

Show that above third-order linear DE can be converted to a first-order linear system of the form

$$\frac{d}{dt}\vec{y}(t) = A(t)\vec{y}(t) + \vec{g}(t),$$

where $A(t)$ is a 3×3 matrix-valued function and $\vec{g}(t)$ is a vector-valued function of dimension 3. **Give explicit formulas for $A(t)$ and $\vec{g}(t)$.**