Math 20D Lecture 10: Numerically Solving D.E.s; P.D.E.s

Thursday, July 28, 2022 4:53 PM

- Final on Saturday July 30th at SOLIS 104; 7 to 10 pm

Numerically solving DEs

Consider DE $\frac{d}{dt}x = f(t,x)$ $\quad x : I \to \mathbb{R}^n$, $f : I \to \mathbb{R}^n$

Say f is cont. diff. and we're given $x(0) = x_0$.

Def: A discretization of the IVP $\begin{cases} \frac{d}{dt}x = f(t,x) \\ x(0) = x_0 \end{cases}$

is a sequence of approximate problems that can be solved (e.g., on a computer). It is consistent if the approx. error goes to 0 as we go further in this sequence.

[The Forward Euler Method]

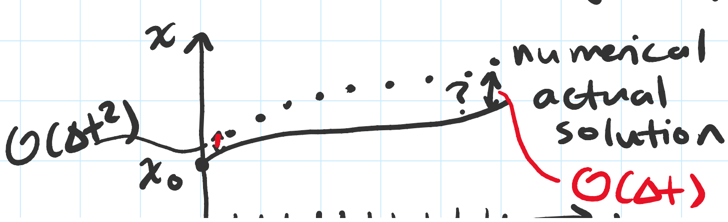
$$x(t+\Delta t) = x(t) + \Delta t\, x'(t) + \mathcal{O}(\Delta t^2)$$
$$= x(t) + \Delta t\, f(t, x(t)) + \mathcal{O}(\Delta t^2)$$

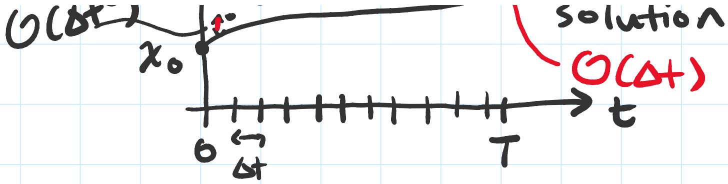
The forward Euler method says ignore

$t_n = n\Delta t \qquad n = 1, 2, 3, \ldots$

$x_n = x(n\Delta t)$

$\rightarrow \quad x_{n+1} = x_n + \Delta t\, f(t_n, x_n), \quad x_0 = x(0) = x_0$





Fix $T > 0$, want to know the solution on $[0,T]$

Number of steps $N = \frac{T}{\Delta t}$ as $\Delta t \to 0$, T fixed, $N \to \infty$

Total error

$$= \begin{pmatrix} \text{number of} \\ \text{timesteps} \end{pmatrix} \times \begin{pmatrix} \text{error per} \\ \text{step} \end{pmatrix}$$

$$= \frac{T}{\Delta t} \times \mathcal{O}(\Delta t^2) = \underbrace{\mathcal{O}(\Delta t)}_{\to 0 \text{ as } \Delta t \to 0}.$$

first-order because global error Δt^1.

<u>Def</u>: A discretization is a reduction of a problem w/ an infinite-dimensional candidate solution space to one with a finite-dim. candidate solution space. It is consistent if the approx error goes to zero as the finite-dimension goes to ∞.

$$\frac{d}{dt} x = f(t,x)$$
$$x(0) = x_0$$

Search for $x \in \underbrace{C^2([0,T])}_{\text{infinite-dim. vector space}}$.

approximation space

$\mathbb{R}^N$ ← number of timesteps

Another 1st-order method: Backward-Euler (BE)

$$x_{n+1} = x_n + \Delta t\, f(t_{n+1}, x_{n+1})$$

algebraic equation to solve for x_{n+1}.

$\Rightarrow$ More expensive to solve than FE.

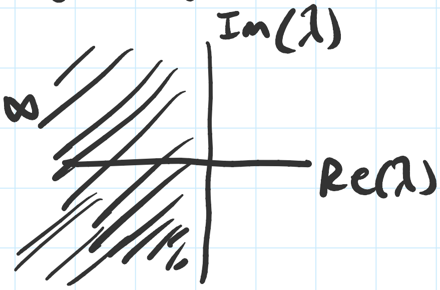
Why would I use this? Stability

(Linear Stability Analysis)

consider
$$\frac{d}{dt} y = \lambda y$$
$$y(0) = y_0$$

True solution $y(t) = y_0 e^{\lambda t}$

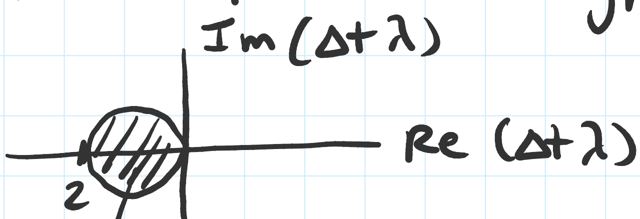
If $\mathrm{Re}(\lambda) < 0$, $y(t) \to 0$ as $t \to \infty$



FE:

$$y_{n+1} = y_n + \Delta t\, \lambda y_n$$
$$= (1 + \Delta t\, \lambda) y_n$$
$$= (1 + \Delta t\, \lambda)^{n+1} y_0$$

$|1 + \Delta t\, \lambda| < 1$ for $y_n \to 0$ as $n \to \infty$



Stability region for FE

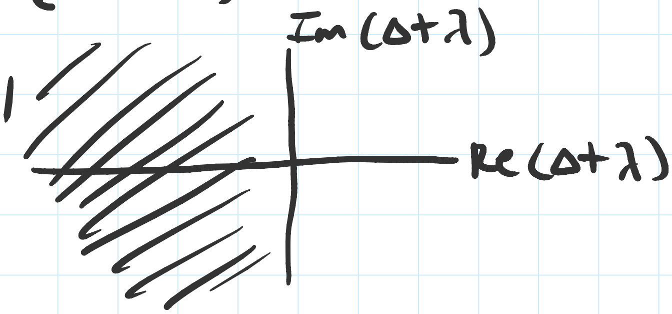
Assume $\lambda \in \mathbb{R}$, $\lambda < 0$, $0 < \Delta t < -\frac{2}{\lambda}$.

restricts timestep

BE: $y_{n+1} = y_n + \Delta t \lambda y_{n+1}$

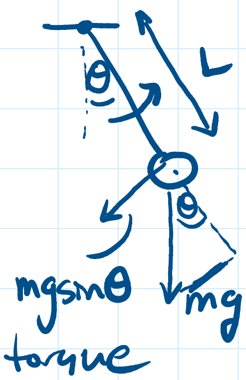
$$y_{n+1} = \frac{y_n}{1-\Delta t\lambda} = \frac{y_0}{(1-\Delta t\lambda)^{n+1}}$$

For stability, $|1-\Delta t\lambda| > 1$



BE is unconditionally stable

Let's consider an ex:



Newton's 2nd law for rotational motion

$$\text{torque} = \begin{pmatrix}\text{moment of}\\ \text{inertia}\end{pmatrix} \times \begin{pmatrix}\text{angular}\\ \text{accel.}\end{pmatrix}$$

$$-mgL\sin\theta = mL^2 \frac{d^2\theta}{dt^2}$$

$$\Rightarrow \underline{\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta} \qquad \left(\begin{array}{l}\text{linearize}\\ \frac{d^2}{dt^2}\theta = -\frac{g}{L}\theta + \cancel{O(\theta^3)}\end{array}\right)$$

Note: energy $E = \frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \frac{g}{L}\cos\theta$ is conserved

$$\frac{dE}{dt} = \frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta\,\frac{d\theta}{dt}$$

$$= \frac{d\theta}{dt}\left(\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta\right) = 0.$$

$$= \frac{d\theta}{dt}\Big(\underbrace{\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta}_{=0}\Big) = 0.$$

Let's solve w/ FE

$$v = \frac{d\theta}{dt}$$

$$\frac{dv}{dt} = -\frac{g}{L}\sin\theta \qquad g = 1 = L$$

FE reads

$$\theta_{n+1} = \theta_n + \Delta t\, v_n$$

$$v_{n+1} = v_n - \Delta t \sin\theta_n$$

BE

$$\theta_{n+1} = \theta_n + \Delta t\, v_{n+1}$$

$$v_{n+1} = v_n - \Delta t \sin\theta_{n+1}$$

Symplectic Euler (SE)

FE in θ: $\quad \theta_{n+1} = \theta_n + \Delta t\, v_n$

BE in v: $\quad v_{n+1} = v_n - \Delta t \sin\theta_{n+1}$

SE doesn't preserve E exactly, but it oscillates about the constant energy surface.

This is the idea "structure-preserving discretization"

SE preserves the symplectic structure exactly. Symplectic structure is more important than energy conservation.

[Partial Differential Equations]

PDEs generalize ODEs to allow derivatives in more than 1 variable. Describe fields evolving in space time:

Maxwell's equations - electromagnetic fields

Schrödinger's equation - quantum mechanics

Einstein's field eqns - curvature of spacetime

Navier-Stokes - fluid dynamics

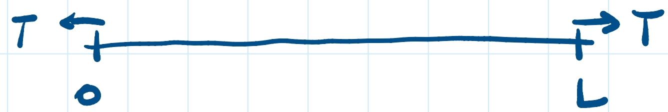
Wave Equation: describes oscillations of some physical quantity propagating through space and time

ex/ light
ex/ gravitational waves

Waves on a String

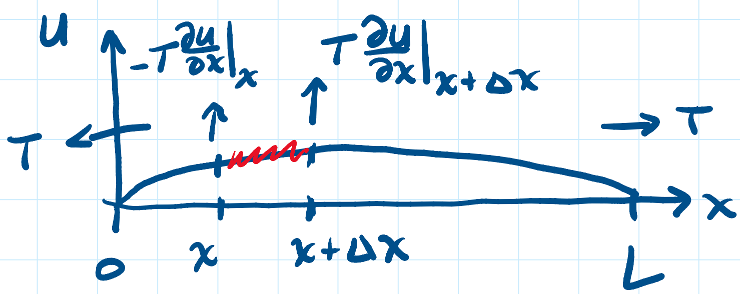
Waves on a String

Consider a string clamped at two ends $x=0$, $x=L$



Suppose the string has linear density ρ (mass/unit length) and tension T

Let $u(t,x)$ denote the displacement of the string at time t and position x.



"Boundary Conditions"
$u(t,0)=0$
$u(t,L)=0$

$F=ma$

$$T\left(\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u}{\partial x}\Big|_{x}\right) = \overbrace{\rho\Delta x}^{m}\frac{\partial^2}{\partial t^2}u(t,x) + O(\Delta x^2)$$

Divide by Δx and let $\Delta x \to 0$

$$T\underbrace{\left(\frac{\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u}{\partial x}}{\Delta x}\right)}_{\frac{\partial^2 u}{\partial x^2}} = \rho\frac{\partial^2}{\partial t^2}u + \underbrace{O(\Delta x)}_{\downarrow\, 0}$$

Wave Equation $T\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\dots}$

Wave Equation $\qquad T\frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad c = \sqrt{T/\rho}$$

c ← speed

BCs $\begin{cases} u(t,0) = 0 \\ u(t,L) = 0 \end{cases}$

ICs $\begin{cases} u(0,x) = u_0(x) \\ \frac{\partial u}{\partial t}(0,x) = v_0(x) \end{cases}$

Solve this using separation of variables

$$u(t,x) = X(x)T(t) \qquad X(0) = 0 = X(L)$$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t) \qquad \frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$$

$$X''(x)T(t) = \frac{1}{c^2}X(x)T''(t)$$

$$\frac{X''(x)}{X(x)} = \underbrace{\sigma}_{\text{const.}} = \frac{1}{c^2}\frac{T''(t)}{T(t)}$$

$\sigma > 0$

$$X''(x) = \sigma X(x) \qquad \cancel{X(x) = c_1 e^{\sqrt{\sigma}x}} + c_2 e^{-\sqrt{\sigma}x}$$

$\sigma = 0 \quad X''(x) = 0 \quad \Rightarrow \quad X(x) = a + bx$

$\sigma = 0 \quad X''(x) = 0 \Rightarrow X(x) = \cancel{C_1 x + C_2}$

$\sigma < 0 \quad \alpha = -\sigma > 0$

$X''(x) = -\alpha X(x)$ $X(x) = C_1 \sin(\sqrt{\alpha}\, x) \cancel{+ C_2 \cos(\sqrt{\alpha}\, x)}$

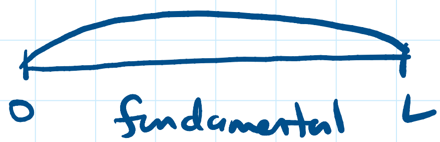
↑ harmonic oscillator

$X(0) = 0$ ✓ $X(L) = 0$

When does $\sqrt{\alpha} L$ satisfy $\sin(\sqrt{\alpha} L) = 0$

$\sqrt{\alpha} L = k\pi \qquad k = 1, 2, 3, \ldots$

$X_k(x) = \underline{C_k} \sin\left(\frac{k\pi}{L} x\right)$

$k=1$


$k=2$


$T''(t) = -\alpha c^2 T(t)$

$\Rightarrow T_k(t) = \beta_k \cos\left(\frac{ck\pi}{L} t\right) + \gamma_k \sin\left(\frac{ck\pi}{L} t\right)$

general solution

linear superposition

$u(t,x) = \sum_{k=1}^{\infty} X_k(x)\, T_k(t)$

$= \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{L} x\right)\left(\beta_k \cos\left(\frac{ck\pi}{\underline{\underline{L}}} t\right) + \gamma_k \sin\left(\frac{ck\pi}{\underline{\underline{L}}} t\right)\right)$

$$u_0(x) = u(0,x) = \sum_{k=1}^{\infty} \beta_k \sin\left(\frac{k\pi}{L} x\right)$$

$$v_0(x) = \frac{\partial u}{\partial t}(0,x) = \sum_{k=1}^{\infty} \frac{ck\pi}{L} \gamma_k \sin\left(\frac{k\pi}{L} x\right)$$

$$\beta_k = \frac{2}{L} \int_0^L \underline{u_0(x)} \sin\left(\frac{k\pi}{L} x\right) dx$$

$$\gamma_k = \frac{2}{ck\pi} \int_0^L \underline{v_0(x)} \sin\left(\frac{k\pi}{L} x\right) dx$$

Fourier sine series.

frequency $\frac{ck\pi}{L} = \sqrt{\frac{T}{\rho}} \frac{k\pi}{L}$