Math 20D Lecture 3: Second-order Linear DEs

Tuesday, July 5, 2022 4:36 PM

- Midterm Friday of Week 3 (July 15) covering the first four lectures of material and homework 1 and 2.
  - The midterm will be available from 12 pm to 11:59 pm on this day on Gradescope. Once you access the midterm, you have 90 minutes to write up your solutions, scan, and submit as a pdf on to Gradescope. Access the exam before 10:29 pm for the full time to work on the exam. You can use an 8.5" x 11" sheet of paper (front and back); no other resources are allowed nor needed.
  - I will release a practice midterm tomorrow. We will have a problem session on Tuesday of Week 3 (July 12) from 3 to 4:20 pm in this room SOLIS 104 where I will go over the solution to the practice midterm problems. However, I highly advise you to do the practice midterm yourself before the problem session.
- Homework 2 will be posted tomorrow.
- Homework 1 is due tomorrow at 11:59 pm on Gradescope.

Chapter 4: Second-Order DEs

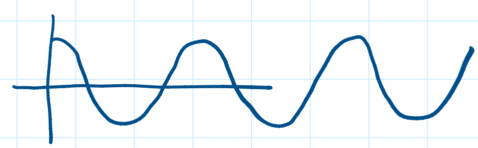
Physical motivation:

Newton's 2nd law: force acting a particle of mass $\overset{\circ}{\hat{m}}$ with position $x(t)$, force given by $F(t, x, \frac{dx}{dt})$.

mass × acceleration = force, $m\frac{d^2x}{dt^2} = F(t, x, \frac{dx}{dt})$

ex/ Harmonic Oscillator

$$\overset{\circ}{\hat{m}}\frac{d^2x}{dt^2} = -\overset{\circ}{\hat{K}}x$$



$$\frac{d^2x}{dt^2} = -\omega^2 x, \qquad \omega = \sqrt{\frac{K}{m}}$$

- solutions $x(t) =$ $\cos(\omega t)$ or $\sin(\omega t)$ $\qquad x: I \rightarrow \mathbb{R}$ ($\mathbb{R} \cup$)
- alternatively, $x: I \rightarrow \mathbb{C}$

$e^{i\omega t}$, $e^{-i\omega t}$ are solutions

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$
$$e^{-i\omega t} = \cos(\omega t) + i\sin(-\omega t)$$
$$= \cos(\omega t) - i\sin(\omega t)$$

$$\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

$$\sin(\omega t) = e^{i\omega t} - e^{-i\omega t}$$

$$e^{-i\omega t} = \cos(\omega t) + i\sin(-\omega t) = \cos(\omega t) - i\sin(\omega t)$$

$$\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

Homogeneous 2nd-order linear DEs w/ constant coefficients (sections 4.2 and 4.3)

$$(*)\quad a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

$f(t)$ homogeneous

constants $a \neq 0$, $b, c \in \mathbb{R}$

Observe that the second derivative of y is a linear combination of its 1st derivative and itself

Try $y(t) = e^{rt}$, r is to be determined

Plug in to the above

$\rightarrow\ ay'' + by' + cy = 0$

$* \rightarrow\ ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$

$e^{rt}(ar^2 + br + c) = 0$

$\neq 0 \quad \Rightarrow \quad \underbrace{ar^2 + br + c}_{\text{Characteristic Polynomial of } (*)} = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \quad y(t) = e^{rt} \text{ is a solution of } (*)$$

ex/ Solve IVP

$$y'' + 7y' + 6y = 0, \quad \underline{y(0) = 1}, \quad \underline{y'(0) = -2}$$

$y(t) = e^{rt}$ plug in

$y'' + 7y' + 6y = 0, \quad \underline{y(0)=1}, \quad \underline{y'(0) = -2}$

$y(t) = e^{rt}$ plug in

$e^{rt}(r^2 + 7r + 6) = 0$

$(r+6)(r+1) = 0 \qquad r = -6, \quad r = -1$

$$y(t) = c_1 e^{-6t} + c_2 e^{-t}$$

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 \\ \oplus \quad -2 &= y'(0) = -6c_1 - c_2 \\ \hline -1 &= -5c_1 \end{aligned} \qquad c_1 = 1/5 \qquad c_2 = 4/5$$

$$y(t) = \frac{1}{5} e^{-6t} + \frac{4}{5} e^{-t} \qquad y(0) = 1 \checkmark$$

$$y'(t) = -\frac{6}{5} e^{-6t} - \frac{4}{5} e^{-t} \qquad y'(0) = -2 \checkmark$$

Thm: Let $a \neq 0$, $b, c \in \mathbb{R}$. Then, there exists a unique solution $y: \mathbb{R} \to \mathbb{R}$ of the IVP

$$(*) \quad ay'' + by' + cy = 0, \qquad \begin{aligned} y(0) &= y_0 \in \mathbb{R} \\ y'(0) &= v_0 \in \mathbb{R} \end{aligned}$$

proof:

$(*)$ can be rewritten as first-order linear system

$$\frac{d}{dt} \vec{x} = A \vec{x} \qquad A \in \mathbb{R}^{2\times 2}$$

$$y' = v, \qquad av' + bv + cy = 0$$

$$v' = -\frac{b}{a} v - \frac{c}{a} y$$

$$\frac{d}{dt} \vec{x} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \vec{x} \qquad \vec{x} = \begin{pmatrix} y \\ v \end{pmatrix}$$

$$\frac{d}{dt} \sim \begin{pmatrix} -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}$$

Consider $ay'' + by' + cy = 0$. The associated char. poly.
$ar^2 + br + c$ has 2 roots:

Two cases:

1 roots are distinct
- a roots are real
- b roots are complex (imaginary part is nonzero)

2 roots are repeated

Case 1: let r_1 and r_2 be the roots, $r_1 \neq r_2$

General solution

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

1a

1b let $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$ $\begin{pmatrix} \alpha = Re(r_1) \\ \beta = Im(r_1) \end{pmatrix}$ ($\beta \neq 0$)

$$y(t) = c_1 e^{(\alpha + i\beta)t} + c_2 e^{(\alpha - i\beta)t}$$

$$= c_1 e^{\alpha t} \underline{e^{i\beta t}} + c_2 e^{\alpha t} \underline{e^{-i\beta t}}$$

replace for

$$= d_1 e^{\alpha t} \cos(\beta t) + d_2 e^{\alpha t} \sin(\beta t)$$

Case 2: Roots $r_1 = r_2$ repeated, call it r.

General solution

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}$$

Proof:

$y_1(t) = e^{rt}$ is a solution of $ay_1'' + by_1' + cy_1 = 0$

Check

$y_2(t) = te^{rt}$ is a solution of $\underline{ay_2'' + by_2' + cy_2} = 0$

$y_2'(t) = e^{rt} + rte^{rt}$

$y_2(t) = te^{rt}$ is a solution of $\underline{ay_2'' + by_2' + cy_2} = 0$

$$y_2'(t) = e^{rt} + rte^{rt}$$

$$y_2''(t) = re^{rt} + re^{rt} + r^2te^{rt}$$

$$ay_2'' + by_2' + cy_2$$
$$= a(re^{rt} + re^{rt} + r^2te^{rt}) + b(e^{rt} + rte^{rt}) + cte^{rt}$$
$$= \underbrace{(ar^2 + br + c)}_{=0}te^{rt} + \cancel{(2ar+b)}^{0}e^{rt}$$
$$= 0$$

$$r = \frac{-b \pm \cancel{\sqrt{b^2-4ac}}^{0}}{2a} = \frac{-b}{2a} \iff 2ar + b = 0$$

Def: Two functions $y_1 : I \to \mathbb{C}$ and $y_2 : I \to \mathbb{C}$ are linearly independent if:

the only solution to $c_1y_1(t) + c_2y_2(t) = 0$ for all $t \in I$ is $c_1 = 0 = c_2$.

(i.e., y_1 and y_2 are not constant multiples of each other)

Theorem:

Let y_1 and y_2 be two linearly independent solutions to the DE $ay'' + by' + cy = 0$ (*)
Then, there exists unique constants c_1 and c_2 such that $y(t) = c_1y_1(t) + c_2y_2(t)$

solves the IVP (*) w/ $y(t_0) = y_0$, $y'(t_0) = v_0$

solves the IVP (*) w/ $y(t_0) = y_0$, $y'(t_0) = v_0$

proof:

Lemma: Let y_1 and y_2 be solutions of (*).
If y_1 and y_2 are linearly independent, then the quantity

$$W(y_1(t), y_2(t)) = y_1(t)\,y_2'(t) - y_1'(t)\,y_2(t) \neq 0 \text{ for all } t$$

Wronskian of y_1 and y_2.

proof: Contrapositive $A \Rightarrow B$

not $B \Rightarrow$ not A

Prove the contrapositive.
Assume $W(y_1(\tau), y_2(\tau)) = 0$ for some τ.

(case 1) $y_1(\tau) \neq 0$. Then, let $\alpha = \dfrac{y_2(\tau)}{y_1(\tau)}$

consider $y(t) = \alpha y_1(t)$

$$y(\tau) = \alpha y_1(\tau) = y_2(\tau)$$

$$y'(\tau) = \frac{y_2(\tau)}{y_1(\tau)} y_1'(\tau) = y_2'(\tau)$$

$y = y_2 \Rightarrow y_2 = \alpha y_1$ linearly dependent.

uniqueness of the IVP above

(case 2): $y_1(\tau) = 0$ but $y_1'(\tau) \neq 0$.

see textbook

(case 3): $y_1(\tau) = 0$ and $y_1'(\tau) = 0$

$y_1(t) = 0$ for all t

$\Rightarrow$ y_1 and y_2 are linearly dependent.

$\square$

y_1 and y_2 are lin. ind. solutions to $(*)$

By linearity

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \quad \text{solves} \quad (*).$$

Initial conditions

$$y_0 = y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0)$$

$$v_0 = y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0)$$

$$\begin{pmatrix} y_0 \\ v_0 \end{pmatrix} = \underbrace{\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}}_{=A,\ \text{Invertible?}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\det A = y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0) \neq 0$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = A^{-1} \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}$$

$$c_1 = \frac{y_0 y_2'(t_0) - v_0 y_2(t_0)}{W(y_1(t_0), y_2(t_0))}$$

$$c_2 = \frac{v_0 y_1(t_0) - y_0 y_1'(t_0)}{W(y_1(t_0), y_2(t_0))} \qquad \square$$

Recalling the cases 1a, 1b, 2 from earlier:

1a: real & distinct roots for char. poly. $r_1 \neq r_2$

$$y_1(t) = e^{r_1 t}$$
$$y_2(t) = e^{r_2 t}$$

$y_2(t) = e^{r_2 t}$

Are they linearly independent?

$$W(y_1(t), y_2(t)) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$
$$= (e^{r_1 t})(r_2 e^{r_2 t}) - (r_1 e^{r_1 t})(e^{r_2 t})$$
$$= \underbrace{(r_2 - r_1)}_{\neq 0} \underbrace{e^{(r_1+r_2)t}}_{\neq 0} \neq 0.$$

$\Rightarrow$ y_1 and y_2 are lin. ind.

$\Rightarrow$ General solution we wrote for case 1a can be used to solve the IVP ✓

1b: Roots are complex $r_1 = \alpha + i\overset{0}{\underset{\#}{\beta}}$

$r_2 = \alpha - i\beta$

$$y_1(t) = e^{\alpha t}\cos(\beta t)$$
$$y_2(t) = e^{\alpha t}\sin(\beta t)$$

$$W(y_1(t), y_2(t)) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$
$$= e^{\alpha t}\cos(\beta t)\left(\alpha e^{\alpha t}\sin(\beta t) + e^{\alpha t}\beta\cos(\beta t)\right)$$
$$- \left(\alpha e^{\alpha t}\cos(\beta t) - e^{\alpha t}\beta\sin(\beta t)\right)e^{\alpha t}\sin(\beta t)$$
$$= e^{2\alpha t}\cancel{\alpha\cos(\beta t)\sin(\beta t)} - e^{2\alpha t}\cancel{\alpha\cos(\beta t)\sin(\beta t)}$$
$$+ e^{2\alpha t}\beta\cos^2(\beta t) + e^{2\alpha t}\beta\sin^2(\beta t)$$
$$= \beta e^{2\alpha t}(\cos^2(\beta t) + \sin^2(\beta t))$$
$$= \beta e^{2\alpha t} \neq 0$$

$$= \underset{\neq 0}{\beta}\ \underset{\neq 0}{e^{2\alpha t}} \neq 0$$

✓

Case 2: Repeated roots $r_1 = r_2$, say $= r$

$$y_1(t) = e^{rt}$$
$$y_2(t) = te^{rt}$$

$$\begin{aligned} W(y_1(t), y_2(t)) &= y_1(t)\, y_2'(t) - y_1'(t)\, y_2(t) \\ &= e^{rt}(e^{rt} + rte^{rt}) \\ &\quad - re^{rt}te^{rt} \\ &= e^{2rt} + \cancel{rte^{2rt}} - \cancel{rte^{2rt}} \\ &\neq 0. \end{aligned}$$

ex/ Solve IVP

$$y'' + 2y' + y = 0 \qquad y(0) = 1, \quad y'(0) = 1$$

characteristic polynomial $r^2 + 2r + 1$

$(r+1)^2$

roots are $r = -1$ (repeated)

$$y_1(t) = e^{-t}$$
$$y_2(t) = te^{-t}$$

General solution

$$\begin{aligned} y(t) &= c_1 y_1(t) + c_2 y_2(t) \\ &= c_1 e^{-t} + c_2 te^{-t} \end{aligned}$$

$1 = y(0) = c_1$

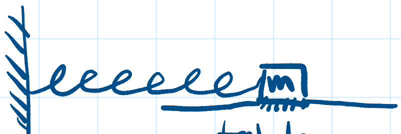
$1 = y'(0) = -c_1 + c_2 \quad \Rightarrow \quad y = e^{-t} + 2te^{-t}$

$\Rightarrow c_2 = 2$

$1 = y'(0) = -c_1 + c_2 \Rightarrow y = \ldots$

$\Rightarrow c_2 = 2$

Ex/ Damped Oscillator



restoring force from spring

$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt}$$

$m > 0$, $k > 0$, $\gamma > 0$ friction constant

characteristic polynomial

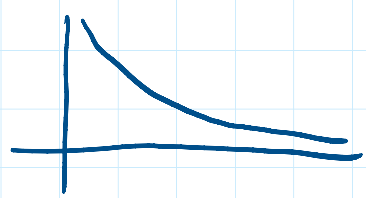
$mx'' + \gamma x' + kx = 0$

$\to mr^2 + \gamma r + k$

roots $\quad r = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$

say friction is large $\quad \gamma^2 >> 4mk$

roots are real.



"over-damped"

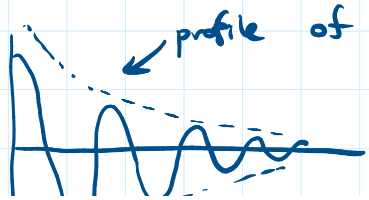
say friction is small $\quad \gamma^2 << 4mk$

roots are complex

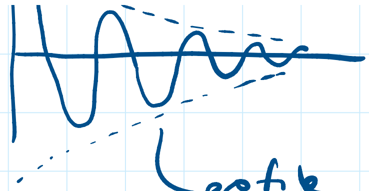
$y_1(t) = e^{-\frac{\gamma}{2m}t}\cos(\omega t)$ $\qquad \omega = \dfrac{\sqrt{\gamma^2 - 4mk}}{2m}$

$y_2(t) = e^{-\frac{\gamma}{2m}t}\sin(\omega t)$

profile of $e^{-\frac{\gamma}{2m}t}$



"under-damped"



profile $- e^{-\frac{\gamma}{2m}t}$

"under-damped"

say friction and other force comparable,

$$\gamma^2 = 4mk$$

repeated roots $\quad r = -\frac{\gamma}{2m}$ "critically-damped"

Principle of Superposition (4.5)

Let y_1 solve $ay_1'' + by_1' + cy_1 = f_1(t)$

Let y_2 solve $ay_2'' + by_2' + cy_2 = f_2(t)$

What does $y(t) = c_1 y_1(t) + c_2 y_2(t)$ solve?

$$\begin{aligned} &ay'' + by' + cy \\ &= ac_1y_1'' + ac_2y_2'' + bc_1y_1' + bc_2y_2' \\ &\quad + cc_1y_1 + cc_2y_2 \\ &= c_1(ay_1'' + by_1' + cy_1) + c_2(ay_2'' + by_2' + cy_2) \\ &= c_1 f_1(t) + c_2 f_2(t) \end{aligned}$$

Now, we want to solve

$$(*)\quad ay'' + by' + cy = f(t)$$

Let y_1 and y_2 be two lin. ind. to the homogeneous problem

$$ay_1'' + by_1' + cy_1 = 0$$
$$ay_2'' + by_2' + cy_2 = 0$$

$$a y_2'' + b y_2' + c y_2 = 0$$

Now, let y_p be a "particular" solution to the inhomogeneous problem,

then, by the principle of superposition above,

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t) \quad \text{solves} \quad (*)$$

__Theorem:__

- Consider the DE $(*)$ $\overset{0}{\overset{\#}{a}}y'' + by' + cy = f(t)$ and suppose $(*)$ has some particular solution $y_p : I \to \mathbb{R}$ and let y_1 and y_2 be lin. ind. solutions to the homogeneous equation.

- Then, there exists a unique solution to the IVP $(*)$ w/ $y(t_0) = y_0$ and $y'(t_0) = v_0$ $(t_0 \in I)$, given by $y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$ $(y : I \to \mathbb{R})$ for the appropriate choice of c_1 & c_2.

__proof:__

$$y_0 = y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) + y_p(t_0)$$

$$v_0 = y'(t_0) = c_1 y_1'(t_0) + c_2 y_2(t_0) + y_p'(t_0)$$

$$\begin{pmatrix} y_0 - y_p(t_0) \\ v_0 - y_p'(t_0) \end{pmatrix} = \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} y_0 - y_p(t_0) \\ v_0 - y_p'(t_0) \end{pmatrix} = \underbrace{\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}}_{\substack{\det \neq 0 \text{ since } y_1 \text{ \& } \\ y_2 \text{ are lin. ind.}}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$\Rightarrow$ can invert

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}^{-1} \begin{pmatrix} y_0 - y_p(t_0) \\ v_0 - y_p'(t_0) \end{pmatrix}$$

By constructing a solution, we've proved existence.
For uniqueness, see homework 2. $\square$

Next time 4.6, variation of parameters
to find particular solutions.