Math 20D Lecture 4: Variation of Parameters and Variable Coefficients

Thursday, July 7, 2022 4:36 PM

- Linhui's discussion sections (A01) on WF 5 to 5:50 pm are now remote (see the announcement on Canvas for the link). This gives you the option of attending remote sections or the in-person sections (A02 and A03).
- Homework 1 solutions will actually be posted tomorrow
- Homework 2 is posted
- Practice Midterm is posted
    - Reminder: There is a problem session next Tuesday July 12 from 3 to 4:20 pm in the same lecture hall where I will go over the solutions to the practice midterm (it will also be podcasted). I'd recommend working on it yourself before the session, so you can verify your work and/or clarify any misunderstandings you have during that session. The problem session is 80 minutes but I shouldn't need the whole time to go over the solutions, so if you have any other questions, you can ask then as well.
- Discuss notation issue from lecture 3

Variation of Parameters (4.6)

Trying to find a particular solution y_p

to $(*)\ ay'' + by' + cy = f(t) \qquad a \neq 0$

Textbook Method: Let y_1, y_2 be lin. ind. solutions to the homogeneous problem

$\Rightarrow c_1 y_1(t) + c_2 y_2(t)$ also solves the homog. problem for any constants $c_1, c_2 \in \mathbb{R}$

Can we construct a particular solution to $(*)$ of the form

$$y_p(t) = v_1(t) y_1(t) + v_2(t) y_2(t)$$

Plug this into $ay'' + by' + cy = f(t)$

$$y_p'(t) = \underline{v_1'(t) y_1(t)} + v_1(t) y_1'(t) + \underline{v_2'(t) y_2(t)} + v_2(t) y_2'(t)$$

assume $v_1' y_1 + v_2' y_2 = 0 \leftarrow$

assume $v_1' y_1 + v_2' y_2 = 0 \leftarrow$

$$y_p'(t) = v_1(t) y_1'(t) + v_2(t) y_2'(t)$$

$$y_p''(t) = v_1'(t) y_1'(t) + v_1(t) y_1''(t) + v_2'(t) y_2'(t) + v_2(t) y_2''(t)$$

$$f(t) = a y_p''(t) + b y_p'(t) + c y_p(t)$$

$$= a \left(\underline{v_1' y_1'} + \underline{v_1 y_1''} + \underline{v_2' y_2'} + \underline{v_2 y_2''} \right) + b \left(\underline{v_1 y_1'} + \underline{v_2 y_2'} \right) + c \left(\underline{v_1 y_1} + \underline{v_2 y_2} \right)$$

$$= v_1 (a y_1'' + b y_1' + c y_1)^{=0} + v_2 (a y_2'' + b y_2' + c y_2)^{=0}$$

$$= a v_1' y_1' + a v_2' y_2'$$

$$\Rightarrow \quad a v_1' y_1' + a v_2' y_2' = f(t)$$

$$v_1' y_1 + v_2' y_2 = 0$$

linear system in the unknowns v_1', v_2'

$$\underbrace{\begin{pmatrix} a y_1' & a y_2' \\ y_1 & y_2 \end{pmatrix}} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} f(t) \\ 0 \end{pmatrix}$$

$$\det A = a y_1' y_2 - a y_2' y_1$$

$$= -a W(y_1(t), y_2(t)) \neq 0$$

$$\Rightarrow \quad v_1'(t) = \frac{-f(t)\,y_2(t)}{a\,W(y_1(t), y_2(t))}$$

$$v_2'(t) = \frac{f(t)\,y_1(t)}{a\,W(y_1(t), y_2(t))}$$

Integrate

$$\left.\begin{aligned} v_1(t) &= \int \frac{-f(t)\,y_2(t)}{a\,W(y_1(t), y_2(t))}\,dt \\ v_2(t) &= \int \frac{+f(t)\,y_1(t)}{a\,W(y_1(t), y_2(t))}\,dt \end{aligned}\right\} \text{Variation of Parameters Formula}$$

Particular solution to (*)

$$y_p(t) = v_1(t)\,y_1(t) + v_2(t)\,y_2(t)$$

General solution to (*)

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$$

ex/ $$ax'' + bx' + cx = f(t)$$

f continuous, assume characteristic polynomial has distinct roots

Step 1: Find two lin. ind. homogeneous solutions

$ax'' + bx' + cx = 0 \qquad x = e^{rt}$

$ar^2 + br + c = 0 \Rightarrow r_1, r_2$ roots $(r_1 \neq r_2)$

$x_1(t) = e^{r_1 t}, \quad x_2(t) = e^{r_2 t}$ ...

$$x_1(t) = e^{r_1 t}, \quad x_2(t) = e^{r_2 t} \quad \text{linearly independent}$$

Step 2: Find the particular solution

$$x_p(t) = v_1(t)x_1(t) + v_2(t)x_2(t)$$

$$v_1(t) = \int \frac{-f(t)\,x_2(t)}{a\,W(x_1(t), x_2(t))}\,dt$$

$$v_2(t) = \int \frac{+f(t)\,x_1(t)}{a\,W(x_1(t), x_2(t))}\,dt$$

$$W(x_1(t), x_2(t)) = x_1(t)x_2'(t) - x_1'(t)x_2(t)$$
$$= (r_2 - r_1)e^{(r_1+r_2)t}$$

$$v_1(t) = \int \frac{-f(t)\,e^{r_2 t}}{a(r_2-r_1)\,e^{(r_1+r_2)t}}\,dt \qquad \frac{1}{e^{(r_1+r_2)t}} = e^{-r_1 t - r_2 t}$$

$$= \frac{-1}{a(r_2-r_1)} \int f(t)\,e^{-r_1 t}\,dt$$

$$v_2(t) = \frac{1}{a(r_2-r_1)} \int f(t)\,e^{-r_2 t}\,dt$$

Different perspective on variation of parameters

$$ax'' + bx' + cx = f(t)$$

$$x'' + \frac{b}{a}x' + \frac{c}{a}x = \frac{1}{a}f(t) \qquad \text{(assume } ar^2+br+c \text{ has distinct roots)}$$

$$x'' + \frac{b}{a}x' + \frac{c}{a}x = \frac{1}{a}f(t)$$

(... has distinct roots)

$$x'' = \underbrace{-\frac{b}{a}x'}_{\text{friction}} \underbrace{- \frac{c}{a}x}_{\text{restoring force}} + \underbrace{\frac{1}{a}f(t)}_{\text{driven force}}$$

Then, infinitesimally, at time s, and a short time Δs, the inhomogeneity f imparts a change in velocity $\Delta v = \frac{1}{a} f(s)\,\Delta s$ $\left(\frac{\Delta v}{\Delta s} = \frac{1}{a}f(s)\right)$

$$a\Delta x_s'' + b\Delta x_s' + c\Delta x_s = 0$$

$$\Delta x_s(s) = 0$$

$$\Delta x_s'(s) = \frac{1}{a}f(s)\,\Delta s$$

Solve it: two lin. ind. solutions

$$x_1(t) = e^{r_1 t}, \qquad x_2(t) = e^{r_2 t}$$

general

$$\Delta x_s(t) = c_1 x_1(t) + c_2 x_2(t)$$
$$= c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Using initial conditions,

$$0 = \Delta x_s(s) = c_1 e^{r_1 s} + c_2 e^{r_2 s}$$

$$\frac{1}{a}f(s)\,\Delta s = \Delta x_s'(s) = r_1 c_1 e^{r_1 s} + r_2 c_2 e^{r_2 s}$$

$$\Rightarrow c_1 = -c_2 e^{(r_2 - r_1)s}$$

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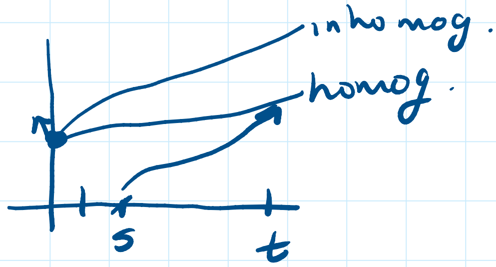
$$\frac{1}{a} f(s)\,\Delta s = -r_1 c_2 e^{(r_2-r_1)s} e^{r_1 s} + r_2 c_2 e^{r_2 s}$$

$$= (r_2 - r_1) e^{r_2 s} c_2$$

$$\Rightarrow \quad c_2 = \frac{1}{a(r_2 - r_1)} f(s) e^{-r_2 s} \Delta s$$

$$c_1 = \frac{-1}{a(r_2-r_1)} f(s) e^{-r_1 s} \Delta s$$



$$\Delta x_s(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$= \frac{-1}{a(r_2-r_1)} f(s) e^{r_1 t} e^{-r_1 s} \Delta s$$

$$+ \frac{1}{a(r_2-r_1)} f(s) e^{r_2 t} e^{-r_2 s} \Delta s$$

$$x(t) = x_h(t) + \sum_{\substack{i \\ \Delta s \to 0}} \Delta x_s(t)$$

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ex/ Solve the IVP

$$x'' + x = \underline{\underline{t}}, \quad x(0) = 2, \quad x'(0) = 1$$

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Find homog. solutions

$x = e^{rt}$ characteristic polynomial $r^2+1 \xrightarrow{\text{roots}} r = \pm i$

$$x_1(t) = \cos(t), \quad x_2(t) = \sin(t)$$

Find particular solution

$$x_p(t) = v_1(t)\,x_1(t) + v_2(t)\,x_2(t)$$

$$v_1(t) = \int \frac{-f(t)\,x_2(t)}{\underbrace{a}_{1}\,W(x_1(t), x_2(t))}\,dt$$

$W(\cos t, \sin t)$
$= \cos t \frac{d}{dt}\sin t - \left(\frac{d}{dt}\cos t\right)\sin t$
$= \cos^2 t + \sin^2 t = 1$

$$= \int -f(t)\,x_2(t)\,dt$$

$$= -\int t\sin t\,dt$$

$$v_2(t) = \int t\cos t\,dt$$

Integrate by parts $\begin{pmatrix} u = t & dv = \cos t\,dt \\ du = 1 & v = \sin t \end{pmatrix}$

$$v_1(t) = t\cos t - \sin t$$

$$v_2(t) = t\sin t + \cos t$$

$\Rightarrow$ Particular solution

$$x_p(t) = v_1 x_1 + v_2 x_2$$

$$= (t\cos t - \sin t)\cos t + (t\sin t + \cos t)\sin t$$

$$= (t\cos t - \sin t)\cos t + (t\sin t + \cos t)\sin t$$

$$= t$$

(satisfies $x_p'' + x_p = t$ ✓)

General Solution

$$x(t) = C_1 x_1(t) + C_2 x_2(t) + x_p(t)$$
$$= C_1 \cos t + C_2 \sin t + t$$

$$2 = x(0) = C_1$$
$$1 = x'(0) = C_2 + 1 \Rightarrow C_1 = 2, \quad C_2 = 0$$

$\Rightarrow$ solution to IVP

$$x(t) = 2\cos t + t.$$

[Variable Coefficient, Linear, 2^{nd}-order equations (4.7)]

Consider equations of the form

$$a_2(t)\, y'' + a_1(t)\, y' + a_0(t)\, y = f(t)$$

Assume $a_2 \neq 0$

$$\Rightarrow \quad y'' + p(t)\, y' + q(t)\, y = g(t)$$

Consider first the homog. case

<u>Theorem</u>:

Let p & q continuous on some interval I
Then, there exists a unique solution
$y: I \to \mathbb{R}$ to the IVP $\quad y'' + p(t)y' + q(t)y = 0 \quad (*)$

$$y(t_0) = y_0 \qquad (t_0 \in I)$$
$$y'(t_0) = v_0$$

<u>proof</u>: Refer to first-order systems.

Write the above system as:

$$v = y'$$
$$v' = y'' = -py' - qy = -pv - qy$$
$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -p(t) & -q(t) \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}, \quad \begin{pmatrix} y(t_0) \\ v(t_0) \end{pmatrix} = \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}$$

$\square$

Suppose we have two homogeneous solutions y_1 and y_2. Are they good enough to solve the IVP for the homogeneous DE? Yes, if they are linearly independent. Furthermore, they also span the solution space.

P, Q cont. on interval I

Let y_1 be the unique solution (on I) to (*) w/ $y_1(t_0)=1$, $y_1'(t_0)=0$ $(t_0 \in I)$

Let y_2 be the unique solution (on I) to (*) w/ $y_2(t_0)=0$, $y_2'(t_0)=1$

$x''+x=0$

$$W(y_1(t_0), y_2(t_0)) = \det\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0$$

Theorem: Let y_1, y_2 be lin. ind. solutions to (*), then the general solution $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is sufficient to solve any IVP associated w/ (*). Furthermore, any other solution of (*) say y_3 is a linear combination of y_1 & y_2.

say y_3 is a linear combination of y_1 & y_2.

(i.e., $\{y_1, y_2\}$ is a basis for solutions to (*))

<u>proof</u>: The first part regarding IVPs is already proved.

Prove "furthermore" part.

Claim: sufficient to prove $\{y_1, y_2, y_3\}$ is linearly dependent.

nontrivial solution to $c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) = 0$

if $c_3 = 0$, $c_1 y_1(t) + c_2 y_2(t) = 0$ contradiction

$$y_3(t) = -\frac{c_1}{c_3} y_1(t) - \frac{c_2}{c_3} y_2(t).$$

proof: assume $\{y_1, y_2, y_3\}$ lin. ind.

$$c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) = 0$$ only has trivial solution

$$c_1 y_1'(t) + c_2 y_2'(t) + c_3 y_3'(t) = 0$$

$$c_1 y_1'(t) + c_2 y_2''(t) + c_3 y_3''(t) = 0$$

$$\underbrace{\begin{pmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1'(t) & y_2'(t) & y_3'(t) \\ y_1''(t) & y_2''(t) & y_3''(t) \end{pmatrix}}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Is A invertible?

$$\det A = W(y_1(t), y_2(t), y_3(t))$$

1, 2, 3

$$\det A = W(y_1(t), y_2(t), y_3(t))$$

$i = 1,2,3$

$$y_i'' = -p(t)y_i' - q(t)y_i$$

after plugging in, you'll see $\det A = 0$

$\Rightarrow$ A not invertible

$\Rightarrow$ nontrivial solutions to this system

$$c_1\cancel{(t)}\,y_1(t) + c_2\cancel{(t)}\,y_2(t) + c_3\cancel{(t)}\,y_3(t) = 0$$

$$\Big(\; A(t) \; \Big) = f(t) \Big(\; A(t_0) \; \Big)$$

$$\begin{pmatrix} t & -t \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{0} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\square$

Inhomogeneous Problem

$(*)$ $y'' + p(t)y' + q(t)y = g(t)$

Suppose we have two lin. ind. homog. solutions y_1 & y_2.

Can we construct a particular solution?

Yes, variation of parameters

Same argument as before

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$$

$$y_p(t) = v_1(t)\,y_1(t) + v_2(t)\,y_2(t)$$

$$v_1(t) = \int \frac{-g(t)\,y_2(t)}{W(y_1(t), y_2(t))}\,dt$$

$$v_2(t) = \int \frac{+g(t)\,y_1(t)}{W(y_1(t), y_2(t))}\,dt$$

<u>Thm</u>: Let p, q, g are continuous on I.

Then, there exists a unique solution to the IVP $(*)$ w/ $y(t_0) = y_0$, $y'(t_0) = v_0$

$(t_0 \in I)$

Furthermore, it can be expressed

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t) \leftarrow$$

for the appropriate choices of c_1, c_2. $\square$

ex/ Cauchy-Euler Equation

$$\underbrace{at^2}_{a_2(t)} y''(t) + \underbrace{bt}_{a_1(t)} y'(t) + \underbrace{c}_{a_0(t)} y(t) = f(t) = 0 \qquad (t > 0)$$

Scaling $t \sim$ seconds $\quad \overset{\sec^2}{\sim}\; t^2 \dfrac{d^2y}{dt^2} \sim \sec^2 \qquad \dfrac{dy}{dt}$

$$\boxed{y(t) = t^r} \qquad y'(t) = rt^{r-1}$$

$$\boxed{y(t) = t^r} \qquad y'(t) = rt^{r-1}$$

$$y''(t) = r(r-1)t^{r-2}$$

$$0 = a\overbrace{t^2 r(r-1)t^{r-2}}^{t^r} + b\overbrace{trt^{r-1}}^{t^r} + ct^r$$

$$= (ar(r-1) + br + c)t^r$$

$$= (ar^2 + (b-a)r + c)t^r \qquad (t>0)$$

$$\Leftrightarrow \quad 0 = ar^2 + (b-a)r + c$$

roots distinct r_1, r_2 $\quad y_1(t) = t^{r_1} \quad y_2(t) = t^{r_2}$
are linearly independent

$$W(y_1(t), y_2(t)) = t^{r_1} r_2 t^{r_2-1} - r_1 t^{r_1-1} t^{r_2}$$

$$= \underbrace{(r_2 - r_1)}_{\neq 0} \underbrace{t^{r_1+r_2-1}}_{\neq 0}$$

ex/ Solve the IVP

$$\underbrace{t^2}_{a_2(t)} y'' - 3ty' + 3y = t^5 = f(t) \qquad y(1) = 0, \quad y'(1) = 0$$

Find homog. solns first

$a = 1, \quad b = -3, \quad c = 3 \qquad y(t) = t^r$

char. poly $\quad ar^2 + (b-a)r + c$

$$r^2 - 4r + 3$$

roots $\quad (r-3)(r-1)$

$\Rightarrow \quad r_1 = 1 \quad r_2 = 3$

roots $(r-3)(r-1)$

$$\Rightarrow r_1 = 1, \quad r_2 = 3$$

$$\Rightarrow y_1(t) = t, \quad y_2(t) = t^3$$

Particular Solution

$$y_p(t) = v_1(t)\,y_1(t) + v_2(t)\,y_2(t)$$

$$v_1(t) = \int \frac{-f(t)\,y_2(t)}{a_2(t)\,W(y_1(t), y_2(t))}\,dt$$

$$= \int \frac{-t^5\,t^3}{t^2\,2t^3}\,dt = \int -\frac{t^3}{2}\,dt$$

$$= \frac{-t^4}{8}$$

$$v_2(t) = \int \frac{f(t)\,y_1(t)}{a_2(t)\,W(y_1(t), y_2(t))}\,dt$$

$$= \int \frac{t^5\,t}{t^2\,2t^3}\,dt = \int \frac{t}{2}\,dt = \frac{t^2}{4}$$

$$y_p(t) = v_1(t)\,y_1(t) + v_2(t)\,y_2(t)$$

$$= \frac{-t^4}{8}\,t + \frac{t^2}{4}\,t^3 = \frac{t^5}{8}$$

General solution $\quad y(1) = 0 \quad y'(1) = 0$

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$$

$$= c_1 t + c_2 t^3 + t^5$$

$$= c_1 t + c_2 t^3 + \frac{t^5}{8}$$

$$0 = y(1) = c_1 + c_2 + 1/8$$

$$0 = y'(1) = c_1 + 3c_2 + 5/8$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1/8 \\ -5/8 \end{pmatrix}$$

Then, invert to get c_1, c_2. $\square$