Math 20D Lecture 6: The Inverse Laplace Transform and Solving IVPs

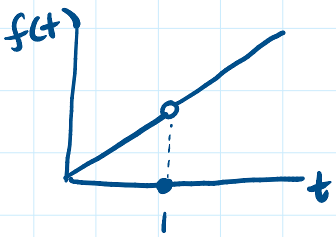
Thursday, July 14, 2022 4:49 PM

- Reminder: The midterm is tomorrow, available to viewed on Gradescope from 12 in the afternoon to 11:59 pm. Once you view the exam, you have 90 minutes to complete and submit your exam (so, make sure to start before 10:29 pm for the full time, and also, make sure to leave about 10 minutes at the end so you have time to scan and upload your exam, as a single PDF file)
- HW2 partial solutions are posted
- HW3 is posted
- MATLAB HW3 due date delayed one day; now due Saturday July 16th at 11:59 pm
- HW4 due date delayed one day; now due Thursday July 28st at 11:59 pm
- Discuss updated course schedule

From last time, $\mathcal{L}(t)(s) = 1/s^2$

Are there other functions, $f(t)$, s.t. $\mathcal{L}(f(t))(s) = 1/s^2$?

yes, for example $f(t) = \begin{cases} t, & t \neq 1 \\ 0, & t = 1 \end{cases}$



$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Def: (Inverse Laplace Transform)

Given a function $F(s)$, if there is a continuous function $f:[0,\infty) \to \mathbb{R}$ s.t. $\mathcal{L}(f) = F$, then we say f is the inverse Laplace transform (ILT) of F, $f(t) = \mathcal{L}^{-1}(F)(t)$.

formula for ILT (not tested)

$$\mathcal{L}^{-1}(F(s))(t) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\,ds$$

γ related to the singularities of F

complex

γ related to the singularities of F)

complex analysis.

ex/ compute $\mathcal{L}^{-1}(F)$ where

1. $F(s) = 1/s^3$ $\qquad$ $\mathcal{L}(t^2) = \dfrac{2}{s^3}$

$\qquad\qquad\qquad\qquad\quad \mathcal{L}\left(\dfrac{t^2}{2}\right) = 1/s^3$

$\Rightarrow \mathcal{L}^{-1}(F)(t) = t^2/2$

2. $F(s) = \dfrac{4}{s^2+16} \Rightarrow \mathcal{L}^{-1}(F)(t) = \sin(4t)$

3. $F(s) = \dfrac{s-5}{s^2-4s+13}$

$$= \frac{s-5}{(s-2)^2+9} = \frac{s-2}{(s-2)^2+9} + \frac{-3}{(s-2)^2+9}$$

$$\Rightarrow \mathcal{L}^{-1}(F)(t) = \underline{e^{2t}\cos(3t)} - \underline{e^{2t}\sin(3t)}$$

<u>Fact</u>: If there exists $f_1, f_2, \ldots$ such that $\mathcal{L}(f_i)(s) = F(s)$ for all i, then at most one of the f_i is continuous, and we define that one to be the ILT of F.

<u>Theorem</u> (Linearity of ILT)

Assume $\mathcal{L}^{-1}(F_1)$ and $\mathcal{L}^{-1}(F_2)$ exist and let $c_1, c_2 \in \mathbb{R}$. Then, the ILT of $c_1F_1 + c_2F_2$ exists and is given by and let $c_1, c_2 \in \mathbb{R}$. Then, the ILT of $c_1F_1 + c_2F_2$ exists and is given by

$$\mathcal{L}^{-1}(c_1F_1 + c_2F_2) = c_1\mathcal{L}^{-1}(F_1) + c_2\mathcal{L}^{-1}(F_2).$$

<u>proof</u>: Follows from linearity of $\mathcal{L}$

Let $f_1 = \mathcal{L}^{-1}(F_1)$ and $f_2 = \mathcal{L}^{-1}(F_2)$.

$$\mathcal{L}(c_1f_1 + c_2f_2) \overset{\text{linearity of } \mathcal{L}}{=} c_1\mathcal{L}(f_1) + c_2\mathcal{L}(f_2)$$

$$= c_1F_1 + c_2F_2.$$

$$\Leftrightarrow c_1f_1 + c_2f_2 = \mathcal{L}^{-1}(c_1F_1 + c_2F_2)$$

$$\Leftrightarrow c_1\mathcal{L}^{-1}(F_1) + c_2\mathcal{L}^{-1}(F_2) = \mathcal{L}^{-1}(c_1F_1 + c_2F_2) \quad \square$$

ex/ Compute

$$\mathcal{L}^{-1}\Big(\underbrace{\frac{3s+2}{s^2+2s+10}}_{F(s)}\Big)$$

$$F(s) = \frac{3s+2}{(s+1)^2+9} = \frac{3(s+1)-1}{(s+1)^2+3^2}$$

$$= 3\,\frac{(s+1)}{(s+1)^2+3^2} - \frac{1}{3}\,\frac{3}{(s+1)^2+3^2}$$

$$\mathcal{L}^{-1}(F)(t) = \mathcal{L}^{-1}\Big(3\,\frac{(s+1)}{(s+1)^2+3^2} - \frac{1}{3}\,\frac{3}{(s+1)^2+3^2}\Big)$$

$$\underset{\text{linearity}}{=} 3\,\mathcal{L}^{-1}\Big(\frac{(s+1)}{(s+1)^2+3^2}\Big) - \frac{1}{3}\mathcal{L}^{-1}\Big(\frac{3}{(s+1)^2+3^2}\Big)$$

$$= 3e^{-t}\cos(3t) - \frac{1}{3}e^{-t}\sin(3t) \quad \square$$

[Partial Fraction Decomposition (PFD)]

A function $\frac{P(s)}{Q(s)}$ where P, Q polynomials,

$\text{degree}(P) < \text{degree}(Q)$, is a rational function.

PFD
$$\frac{P(s)}{Q(s)} = \frac{\cdots}{(s-r_1)} + \frac{\cdots}{(s-r_2)} + \cdots + \frac{\cdots}{(s-r_3)}$$
$$+ \frac{\cdots}{s^2+\cdots} + \frac{\cdots}{s^2+\cdots}$$

3 cases:

Distinct Linear Factors

Suppose Q is a degree n polynomial w/ n distinct real roots $r_1, \dots, r_n$

$$Q(s) = A(s-r_1)(s-r_2)\cdots(s-r_n)$$

Then, $$\frac{P(s)}{Q(s)} = \frac{c_1}{s-r_1} + \cdots + \frac{c_n}{s-r_n}$$
for suitable choices of $c_i \in \mathbb{R}$.

ex/ Compute $\mathcal{L}^{-1}(F)$, where $F(s) = \frac{s+1}{(s-1)(s+2)}$

$$\frac{s+1}{(s-1)(s+2)} = \frac{P(s)}{Q(s)} = \frac{c_1}{s-1} + \frac{c_2}{s+2}$$

$$s+1 = c_1(s+2) + c_2(s-1)$$

$s=1$ $\quad 2 = 3c_1 \quad \Rightarrow c_1 = 2/3$

$s=1 \quad 2 = 3c_1 \quad \Rightarrow c_1 = 2/3$

$s=-2 \quad -1 = -3c_2 \quad \Rightarrow c_2 = 1/3$

$$F(s) = \frac{2/3}{s-1} + \frac{1/3}{s+2} \qquad \mathcal{L}(e^t) = \frac{1}{s-1}, \quad \mathcal{L}(e^{-2t}) = \frac{1}{s+2}$$

$$\Rightarrow \mathcal{L}^{-1}(F)(t) = \frac{2}{3}e^t + \frac{1}{3}e^{-2t} \quad \square$$

[Repeated Factors]

Suppose Q has a factor of the form $(s-r)^m$, $m \geq 2$. Then, the PFD of

$$\frac{P(s)}{Q(s)} = \left(\text{other terms corresponding to other factors}\right) + \frac{c_1}{s-r} + \dots + \frac{c_m}{(s-r)^m}$$

ex/ Compute ILT of

$$F(s) = \frac{s+2 \leftarrow P}{(s-2)^2(s-1) \leftarrow Q}$$

$$= \underbrace{\frac{c_1}{s-1}}_{\text{distinct}} + \underbrace{\frac{d_1}{s-2} + \frac{d_2}{(s-2)^2}}_{\text{repeated roots}}$$

$$s+2 = c_1(s-2)^2 + d_1(s-1)(s-2) + d_2(s-1)$$

$s=1 \quad 3 = c_1$ ✓

$s=2 \quad 4 = d_2$ ✓

$s=2 \quad 4 = d_2$ ✓

$s=0 \quad 2 = 4c_1 + 2d_1 - d_2$

$d_1 = 1 - 2c_1 + \frac{1}{2} d_2 = 1 - 6 + 2 = -3$

$$F(s) = \frac{c_1}{s-1} + \frac{d_1}{s-2} + \frac{d_2}{(s-2)^2} \qquad \frac{d}{ds}\mathcal{L}(f)(s) = (-1)\mathcal{L}(tf)(s)$$

$$\mathcal{L}^{-1}(F)(t) = c_1 e^t + d_1 e^{2t} + d_2 t e^{2t}$$

[Unreducible Quadratic Factors]

Suppose $(s-a)^2 + b^2$ is a factor of $Q(s)$ which can't be reduced in $\mathbb{R}$. Let $((s-a)^2+b^2)^m$ be the highest power of the factor appearing in Q

Then, the PFD of

$$\frac{P(s)}{Q(s)} = \left(\begin{matrix}\text{terms from} \\ \text{other cases}\end{matrix}\right)$$

$$+ \frac{c_1 s + d_1}{(s-a)^2+b^2} + \cdots + \frac{c_m s + d_m}{((s-a)^2+b^2)^m}$$

ex/ Compute ILT of

$$F(s) = \frac{s}{(s^2-2s+5)(s+1)}$$

$b^2 - 4ac = 4 - 20 = -16 < 0$ complex roots

$$b^2-4ac = 4-20 = -16 < 0 \quad \text{complex roots}$$

$$F(s) = \frac{s}{((s-1)^2+2^2)(s+1)}$$

$$= \frac{c_1}{s+1} + \frac{d_1 s + d_2}{((s-1)^2+2^2)}$$

$$s = c_1((s-1)^2+4) + (d_1 s + d_2)(s+1)$$

$s=-1$ $\quad -1 = 8c_1 \quad c_1 = -1/8$

$s=1$ $\quad 1 = 4c_1 + 2d_1 + 2d_2$

$s=0$ $\quad 0 = 5c_1 + d_2$

$$d_2 = -5c_1 = 5/8$$

$$d_1 = \frac{1}{2} - 2c_1 - d_2$$

$$= \frac{1}{2} + \frac{2}{8} - \frac{5}{8} = 1/8$$

$$F(s) = \frac{1}{8}\left(\frac{-1}{s+1} + \frac{s-1+6}{((s-1)^2+2^2)}\right)$$

$$= \frac{1}{8}\left(\frac{-1}{s+1} + \frac{s-1}{(s-1)^2+2^2} + 3\cdot\frac{2}{(s-1)^2+2^2}\right)$$

$$\mathcal{L}^{-1}(F)(t) = \frac{1}{8}\left(-\underline{\underline{e}}^{-t} + \underline{\underline{e}}^{t}\cos(2t) + 3\,\underline{\underline{e}}^{t}\sin(2t)\right)$$

Solving IVPs (7.5)

$$\left(\mathcal{L}(y)(s) = \tilde{y}(s) \right)$$

$$\text{IVP} \xrightarrow{①} \text{LT} \xrightarrow{②} \text{Algebra} \xrightarrow{③} \text{ILT} \rightarrow \text{Solution}$$

ex/ Solve the IVP

$$y'' - y' + y = e^{-t}, \quad y(0) = 1, \quad y'(0) = 1$$

① Take the LT of both sides of DE

$$\mathcal{L}(y'' - y' + y)(s) = \mathcal{L}(e^{-t})(s)$$

$$\underbrace{\mathcal{L}(y'')(s)}_{s^2\tilde{y}(s) - sy(0) - y'(0)} - \underbrace{\mathcal{L}(y')(s)}_{s\tilde{y}(s) - y(0)} + \mathcal{L}(y)(s) = \frac{1}{s+1}$$

$$s^2\tilde{y}(s) - sy(0) - y'(0) - \left(s\tilde{y}(s) - y(0)\right) + \tilde{y}(s) = \frac{1}{s+1}$$

$$s^2\tilde{y}(s) - s \cancel{-1} - s\tilde{y}(s) \cancel{+1} + \tilde{y}(s) = \frac{1}{s+1}$$

② Algebra. Solve for $\tilde{y}(s)$

$$(s^2 - s + 1)\tilde{y}(s) = \frac{1}{s+1} + s = \frac{s^2+s+1}{s+1}$$

$$\Rightarrow \tilde{y}(s) = \frac{s^2+s+1}{(s+1)(s^2-s+1)}$$

$b^2 - 4ac$
$= 1 - 4 < 0$

$$\text{PFD} \quad \tilde{y}(s) = \frac{c_1}{s+1} + \frac{c_2 s + d}{s^2 - s + 1}$$

$s^2 - s + 1 \rightarrow (s - \frac{1}{2})^2 + \frac{3}{4}$

$$s^2 + s + 1 = c_1(s^2 - s + 1) + (c_2 s + d)(s+1)$$

$s = -1, 0, 1$

$\Rightarrow c_1 = 1/3, \quad d = 2/3, \quad c_2 = 2$

$$\Rightarrow c_1 = 1/3, \quad d = 2/3, \quad c_2 = 2/3$$

$$\tilde{y}(s) = \frac{1/3}{s+1} + \frac{2}{3}\frac{(s+1)}{(s-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$$

$$= \frac{1/3}{s+1} + \frac{2}{3}\left(\frac{s-1/2}{\underbrace{(s-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}} + \sqrt{3}\,\frac{\sqrt{3}/2}{(s-\frac{1}{2})^2 + \frac{\sqrt{3}}{2}}\right)$$

$$y(t) = \frac{1}{3}e^{-t} + \frac{2}{3}e^{\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{2}{\sqrt{3}}e^{\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right) \quad \square$$

Using the Laplace transform to solve variable coefficient equations

$$y'' + p(t)y' + q(t)y = g(t)$$

$$y(0) = y_0 \leftarrow$$

$$y'(0) = V_0 \leftarrow$$

p, q are polynomials (one of degree ≥ 1)

claim: The LT of this DE is a DE for $\underline{\underline{\tilde{y}(s)}}$

ex/ $\mathcal{L}\left(y'' + ty = 0\right)$

$$s^2\tilde{y}(s) - s\underset{\underset{y_0}{\shortparallel}}{y(0)} - \underset{\underset{V_0}{\shortparallel}}{y'(0)} + \underbrace{\mathcal{L}(ty)(s)}_{(-1)\frac{d}{ds}\tilde{y}(s)} = 0$$

$v_0 \quad (-1)\frac{d}{ds}\tilde{y}(s)$

$\Rightarrow \; -\tilde{y}'(s) - s y_0 - v_0 + s^2 \tilde{y}(s) = 0$

What are the BCs for $\tilde{y}$?

Theorem:

Let $f : [0,\infty) \to \mathbb{R}$ be p.w. continuous and of exponential order α. Then,

$$\underline{\lim_{s\to\infty} \mathcal{L}(f)(s) = 0} \qquad \text{“} \underline{\tilde{f}(\infty) = 0} \text{”}$$

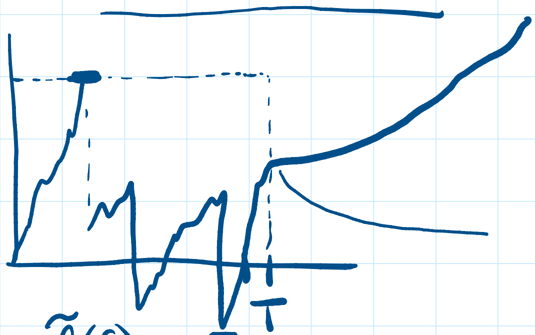
$$\left(\text{corollary} \quad \lim_{s\to\infty} \underline{\frac{d^n}{ds^n}\mathcal{L}(f)(s) = 0}\right)$$

proof:

There exists $T \geq 0$, $M \geq 0$ s.t. $|f(t)| \leq M e^{\alpha t}$, $t > T$

There exists $K \geq 0$, $\tilde{\alpha} \in \mathbb{R}$

s.t. $\underline{|f(t)| \leq K e^{\tilde{\alpha} t}}$ for all $t \geq 0$



$\alpha < 0, \; \tilde{\alpha} = 0$

$\alpha > 0, \; \tilde{\alpha} = \alpha$

$\tilde{\alpha} = \max\{0, \alpha\}$

$K = \max_{[0,T]} |f(t)|$

$\vee\!\mid$
0

$$\Bigg| \overbrace{\int_0^\infty e^{-st} f(t)\, dt}^{\tilde{f}(s)} \Bigg|$$

$\infty \qquad M$

$$\left|\int_0^\infty e^{-st} f(t)\,dt\right|$$

$$\lim_{s\to\infty} \leq \int_0^\infty e^{-st}|f(t)|\,dt \leq \int_0^\infty e^{-st} K e^{\tilde{\alpha} t}\,dt$$

$$= K\int_0^\infty e^{-(s-\tilde{\alpha})t}\,dt \overset{s>\tilde{\alpha}}{=} K\frac{1}{s-\tilde{\alpha}} \longrightarrow 0 \text{ as } s\to\infty$$

$\square$

Solve IVP

ex/ $y'' + 2ty' - 4y = 1, \quad y(0)=0, \quad y'(0)=0$

Take LT of both sides

$$\mathcal{L}(y'')(s) + 2\mathcal{L}(ty')(s) - 4\mathcal{L}(y)(s) = \mathcal{L}(1)(s)$$

$$\left(s^2\tilde{y}(s) - s\cancel{y(0)} - \cancel{y'(0)}\right) - 2\frac{d}{ds}\mathcal{L}(y')(s) - 4\tilde{y}(s) = \frac{1}{s}$$

$$s^2\tilde{y}(s) - 2\frac{d}{ds}\left(s\tilde{y}(s) - \cancel{y(0)}\right) - 4\tilde{y}(s) = \frac{1}{s}$$

$$\underline{s^2\tilde{y}(s)} - \underline{2\tilde{y}(s)} - 2s\tilde{y}'(s) - \underline{4\tilde{y}(s)} = 1/s$$

$$\tilde{y}'(s) + \underbrace{\left(\frac{-s}{2} + \frac{3}{s}\right)}\tilde{y}(s) = -1/2s^2$$

$$\mu(s) = \exp\left(\int\left(-\frac{s}{2} + \frac{3}{s}\right)ds\right)$$

$$= \exp\left(-\frac{s^2}{4} + 3\ln(s)\right)$$

$$= e^{-s^2/4}s^3$$

$$\frac{d}{ds}\left(s^3 e^{-s^2/4}\tilde{y}(s)\right) = -\frac{1}{2}s e^{-s^2/4}$$

$u = -s^2/4$

$du = -\frac{1}{2}s\,ds$

Integrate

$$s^3 e^{-s^2/4}\tilde{y}(s) = e^{-s^2/4} + C$$

integrate

$$s^3 e^{-s^2/4}\,\tilde{y}(s) = e^{-s^2/4} + C$$

$$\Rightarrow \tilde{y}(s) = \underbrace{\frac{1}{s^3}}_{\downarrow (s\to\infty)\ 0} + \underset{\overset{\shortparallel}{0}}{C}\,\frac{e^{s^2/4}}{\underbrace{s^3}_{\to\infty}}$$

$C=0$ by the fact that $\lim_{s\to\infty} \tilde{y}(s) = 0$.

$$\Rightarrow \tilde{y}(s) = 1/s^3$$

$$\Rightarrow y(t) = t^2/2.$$

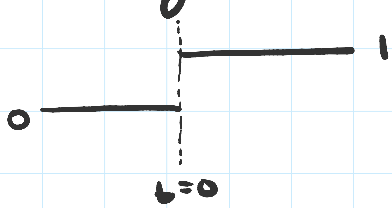
$\square$

LT of piecewise continuous functions

Notation:

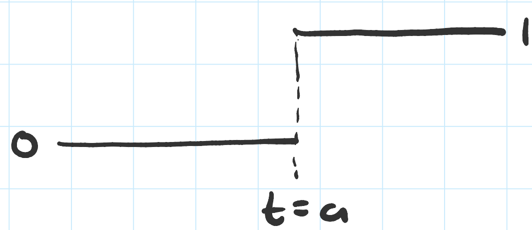
<u>Def</u>: The unit step function (or Heaviside function) $u(t)$ is defined by

$$u(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}$$



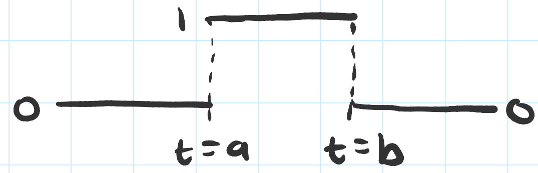
For $a\in\mathbb{R}$,

$$u(t-a) = \begin{cases} 0, & t<a \\ 1, & t>a \end{cases}$$



The window function (or the indicator function) for an interval (a,b) is

$$\chi_{(a,b)}(t) = \begin{cases} 0, & t \notin (a,b) \\ 1, & t \in (a,b) \end{cases}$$



$$\chi_{(a,b)}(t) = u(t-a) - u(t-b) \quad (a < b)$$

Using these, any piecewise function can be expressed in a single expression

ex/ $f : [0, \infty) \to \mathbb{R}$

$$f(t) = \begin{cases} t, & 0 < t < \pi \\ \sin(t), & \pi < t < 2\pi \\ e^t, & t > 2\pi \end{cases}$$

$$f(t) = t\,\chi_{(0,\pi)}(t) + \sin(t)\,\chi_{(\pi,2\pi)}(t) + e^t u(t-2\pi)$$

Prop: Let $a \geq 0$. Then, $\mathcal{L}(u(t-a))(s) = \dfrac{e^{-as}}{s}$ $(s > 0)$

Proof:

$$\mathcal{L}(u(t-a))(s) = \int_0^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_a^\infty e^{-st}\, dt = \left.\frac{e^{-st}}{-s}\right|_{t=a}^{t=\infty}$$

$$= \int_a^{\infty} e^{-st}\,dt = \frac{e^{-st}}{-s}\Big|_{t=a}$$

$$= e^{-as}/s \quad (s>0)$$

$\square$

<u>Prop</u>: Let $f:[0,\infty)\to\mathbb{R}$ be s.t. its LT exists for $s>\alpha$. Then, for $a \geq 0$,

$$\mathcal{L}(f(t-a)\,u(t-a))(s) = e^{-as}\mathcal{L}(f)(s) \qquad s>\alpha$$

<u>Proof</u>: $\mathcal{L}^{-1}\left(e^{-as}\mathcal{L}(f)(s)\right) = f(t-a)\,u(t-a)$

$$\mathcal{L}(f(t-a)u(t-a))(s)$$

$$= \int_0^{\infty} e^{-st} f(t-a)\,u(t-a)\,dt$$

$\tau = t-a \qquad t = \tau + a$

$$= \int_a^{\infty} e^{-st} f(t-a)\,dt = \int_0^{\infty} e^{-s(\tau+a)} f(\tau)\,d\tau$$

$$= e^{-as}\int_0^{\infty} e^{-s\tau} f(\tau)\,d\tau$$

$$= e^{-as}\mathcal{L}(f)(s)$$

$\square$

Usually easier

$$\mathcal{L}(g(t)u(t-a)) = e^{-as}\mathcal{L}(g(t+a))(s)$$

ex Solve $\frac{dx}{dt} = f(t)$, $x(0) = 0$

$$f(t) = \begin{cases} 1, & 0<t<1 \\ 2, & 1<t<\infty \end{cases}$$

Take LT of both sides, note

$f(t) = \chi_{\ldots}(t) + 2u(t-1)$

Take LT of both sides, note

$$f(t) = \chi_{(0,1)}(t) + 2u(t-1)$$
$$= u(t-0) - u(t-1) + 2u(t-1)$$
$$= 1 + u(t-1)$$

$$\mathcal{L}\left(\frac{dx}{dt}\right)(s) = \mathcal{L}(f)(s)$$

$$s\tilde{x}(s) - \cancel{x(0)} = \frac{1}{s} + \frac{e^{-s}}{s}$$

$$\tilde{x}(s) = \frac{1}{s^2} + \frac{e^{-s}}{s^2}$$

$$x(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right)(t) + \mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2}\right)(t)$$

$$= t + \underbrace{}_{}$$

$(t-1)\,u(t-1)$

$$\mathcal{L}^{-1}\left(e^{-as}\mathcal{L}(f)(s)\right) = f(t-a)\,u(t-a)$$

$\mathcal{L}(f)(s) = 1/s^2$

$f(t) = \mathcal{L}^{-1}(1/s^2)(t) = t$

$$= t + (t-1)\,u(t-1)$$

$$= \begin{cases} t, & 0 < t < 1 \\ 2t-1, & t > 1 \end{cases}$$