Math 20D Lecture 7: Convolution and the Dirac Delta Distribution

Tuesday, July 19, 2022 4:58 PM

- Midterm Solutions posted
- Homework 3 due tomorrow (Wednesday) at 11:59 pm
- Homework 4 will be posted tomorrow. Due Thursday July 28th at 11:59 pm
- Practice Final will be posted tomorrow.

ex/ $x''(t) + 4x(t) = g(t)$ $\quad x(0) = 0 = x'(0)$

where $g(t) = \begin{cases} 1, & 0<t<1 \\ -1, & 1<t<2 \\ 0, & t>2 \end{cases}$

$x'' = -4x + g(t)$ "a priori estimate": based on the regularity of the inhomogeneity, I can tell the regularity of the solution, <u>without</u> knowing the solution.

$x'' + 4x = g(t)$

Take LT of both sides

$\Rightarrow s^2\tilde{x}(s) - s\cancel{x(0)} - \cancel{x'(0)} + 4\tilde{x}(s) = \mathcal{L}(g(t))(s)$

$(s^2+4)\tilde{x}(s)$

$= \mathcal{L}(g(t))(s)$

$g(t) = 1 - u(t-1) + (-1)(u(t-1) - u(t-2)) + \cancel{0 \cdot u(t-2)}$
$= 1 - 2u(t-1) + u(t-2)$

$\mathcal{L}(g(t))(s) = \frac{1}{s} - 2\frac{e^{-s}}{s} + \frac{e^{-2s}}{s}$

$(s^2+4)\tilde{x}(s) = \frac{1}{s} - 2\frac{e^{-s}}{s} + \frac{e^{-2s}}{s}$

$\tilde{x}(s) = \frac{1}{s(s^2+4)} - 2\frac{e^{-s}}{s(s^2+4)} + \frac{e^{-2s}}{s(s^2+4)}$

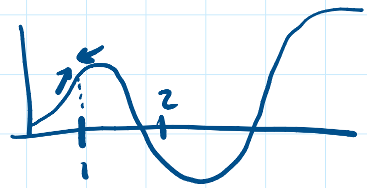
$\tilde{f}(s) = \frac{1}{s(s^2+4)} = \frac{1}{4}\cdot\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}$

$$\tilde{f}(s) = \frac{1}{s(s^2+4)} = \frac{1}{4}\cdot\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}$$

$$\Rightarrow \tilde{x}(s) = \tilde{f}(s) - 2e^{-s}\tilde{f}(s) + e^{-2s}\tilde{f}(s)$$

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}(\tilde{x}(s))(t) \\ &= \mathcal{L}^{-1}(\tilde{f}(s)) - 2\mathcal{L}^{-1}(e^{-s}\tilde{f}(s)) + \mathcal{L}^{-1}(e^{-2s}\tilde{f}(s)) \\ &= f(t) - 2u(t-1)f(t-1) + u(t-2)f(t-2) \end{aligned}$$

$$f(t) = \frac{1}{4} - \frac{1}{4}\cos(2t)$$



Check that x is continuous.

It suffices to check at $t=1$ & $t=2$.

For $t=1$,

$$\lim_{t\to 1^-} x(t) = \lim_{t\to 1} f(t) = \lim_{t\to 1}\left(\frac{1}{4} - \frac{1}{4}\cos(2t)\right) = \frac{1}{4} - \frac{1}{4}\cos(2).$$

$$\begin{aligned} \lim_{t\to 1^+} x(t) &= \lim_{t\to 1}\left(f(t) - 2f(t-1)\right) \\ &= \lim_{t\to 1}\left(\frac{1}{4} - \frac{1}{4}\cos(2t) - 2\left(\frac{1}{4} - \frac{1}{4}\cos(2(t-1))\right)\right) \\ &= \frac{1}{4} - \frac{1}{4}\cos(2). \quad ✓ \end{aligned}$$

(similar for $t=2$)

For $\frac{dx}{dt}$, $\quad x(t) = f(t) - 2u(t-1)f(t-1) + u(t-2)f(t-2)$

$$f'(t) = \frac{d}{dt}\left(\frac{1}{4} - \frac{1}{4}\cos(2t)\right) = \frac{1}{2}\sin(2t)$$

$$x'(t) = f'(t) - 2u(t-1)f'(t-1) + u(t-2)f'(t-2)$$

$$x'(t) = f'(t) - 2u(t-1)f'(t-1) + u(t-2)f'(t-2)$$

$t=1$

$$\lim_{t\to 1^-} x'(t) = \lim_{t\to 1^-} f'(t) = \lim_{t\to 1} \frac{1}{2}\sin(2t) = \frac{\sin(2)}{2}.$$

$$\lim_{t\to 1^+} x'(t) = \lim_{t\to 1^+} \left(f'(t) - 2f'(t-1)\right)$$

$$= \lim_{t\to 1}\left(\frac{1}{2}\sin(2t) - \sin(2(t-1))\right) = \frac{\sin(2)}{2} \checkmark$$

Convolution (7.8)

consider the IVP $\qquad a \neq 0$

$$ay'' + by' + cy = g(t), \quad y(0) = 0 = y'(0)$$

$\overset{LT}{\Rightarrow}$ $(as^2 + bs + c)\tilde{y}(s) = \tilde{g}(s)$

$$\tilde{y}(s) = \left(\frac{1}{as^2+bs+c}\right)\tilde{g}(s)$$

How do I compute the ILT of $\tilde{h}(s)\tilde{g}(s)$?

Is $\mathcal{L}^{-1}(\tilde{h}(s)\tilde{g}(s)) = h(t)g(t)$? **NO**

<u>Def:</u> Let f, g be piecewise cont. $[0,\infty) \to \mathbb{R}$.

Then, the convolution of f and g,

$f * g : [0,\infty) \longrightarrow \mathbb{R}$, is defined by

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

ex/ $f(t) = t^2, \quad g(t) = t^3$

$$(f*g)(t) = \int_0^t (t-\tau)^2 \tau^3 \, d\tau$$

$$= \int_0^t (t^2 - 2t\tau + \tau^2)\tau^3 \, d\tau$$

$$= \int_0^t (t^2\tau^3 - 2t\tau^4 + \tau^5) \, d\tau$$

$$= \left(\frac{t^2\tau^4}{4} - \frac{2t\tau^5}{5} + \frac{\tau^6}{6}\right)\Bigg|_{\tau=0}^{\tau=t}$$

$$= t^6\left(\frac{1}{4} - \frac{1}{5} + \frac{1}{6}\right) = \frac{13}{60}t^6$$

Properties of $*$

Let $f, g, h : [0,\infty) \to \mathbb{R}$ be piecewise cont. and let $c_1, c_2 \in \mathbb{R}$. Then

(linearity) $f * (c_1 g + c_2 h)$
$= c_1 (f*g) + c_2 (f*h)$

(commutavity) $f * g = g * f$

(associative) $f * (g*h) = (f*g)*h$

<u>Proof</u>: linearity: follows from linearity of the integral

associativity: changing order of integration

commutavity

$$(f*g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

$\rho = t - \tau \qquad d\rho = -d\tau$

$$= -\int^{0} f(\rho)\, g(t-\rho)\, d\rho$$

at $\tau = 0$, $\rho = t$

$$= -\int_t^0 f(\rho)g(t-\rho)\,d\rho \qquad \begin{array}{l} \text{at } \tau=0,\ \rho=t \\ \text{at } \tau=t,\ \rho=0 \end{array}$$

$$= \int_0^t f(\rho)g(t-\rho)\,d\rho = (g * f)(t)$$

$\square$

Theorem:

Let $f, g : [0,\infty) \to \mathbb{R}$ be p.w. cont. and of exponential order α.

Then, $$\mathcal{L}(f * g)(s) = \tilde{f}(s)\tilde{g}(s)$$

i.e., $$\mathcal{L}^{-1}(\tilde{f}(s)\tilde{g}(s))(t) = (f * g)(t)$$

proof:

$\mathcal{L}(f * g)(s)$

$$= \int_0^\infty e^{-st}(f * g)(t)\,dt$$

$$= \int_0^\infty e^{-st}\left(\int_0^t f(t-\tau)g(\tau)\,d\tau\right)dt$$

$$= \int_0^\infty e^{-st}\left(\int_0^\infty u(t-\tau)f(t-\tau)g(\tau)\,d\tau\right)dt$$

$$= \int_0^\infty g(\tau)\left(\underbrace{\int_0^\infty e^{-st}u(t-\tau)f(t-\tau)\,dt}_{e^{-s\tau}\tilde{f}(s)}\right)d\tau$$

$$= \tilde{f}(s)\int_0^\infty e^{-s\tau}g(\tau)\,d\tau = \tilde{f}(s)\tilde{g}(s)$$

$\square$

Solving IVPs using convolution

Solving IVPs using convolution

Consider the IVP

$$ay'' + by' + cy = g(t) \qquad y(0) = y_0$$
$$y'(0) = v_0$$

split solution $\quad y(t) = y_h(t) + y_p(t)$

y_h solves $\quad ay_h'' + by_h' + cy_h = 0, \qquad y_h(0) = y_0$
$\qquad y_h'(0) = v_0$

$\hookrightarrow$ solve ✓

y_p solves $\quad ay_p'' + by_p' + cy_p = g(t) \qquad y_p(0) = 0$
$\qquad y_p'(0) = 0$

$\hookrightarrow$ LT

$$(as^2 + bs + c)\tilde{y}_p(s) = \tilde{g}(s)$$

$$\tilde{y}_p(s) = \underbrace{\frac{1}{as^2 + bs + c}}\tilde{g}(s)$$

define $\quad \tilde{h}(s) = \dfrac{\tilde{y}_p(s)}{\tilde{g}(s)} = \dfrac{1}{as^2 + bs + c}$

the "transfer function" of the system

$$\tilde{y}_p = \tilde{h}\tilde{g}$$

$$y_p(t) = \mathcal{L}^{-1}(\tilde{y}_p)(t)$$
$$= \mathcal{L}^{-1}(\tilde{h}\tilde{g})(t) = \underline{(h * g)(t)}$$

$\Rightarrow$ Full solution $\qquad h(t) = \mathcal{L}^{-1}(\tilde{h}(s))(t)$
$$= \mathcal{L}^{-1}\left(\frac{1}{\quad}\right)(t)$$

$\Rightarrow$ Full solution

$$y(t) = y_h(t) + y_p(t)$$
$$= y_h(t) + \int_0^t h(t-\tau)g(\tau)\,d\tau$$

$$h(t) = \mathcal{L}^{-1}(\tilde{h}(s))(t) = \mathcal{L}^{-1}\left(\frac{1}{as^2+bs+c}\right)(t)$$

Claim:
$h(t)$ satisfies

$$ah'' + bh' + ch = 0, \quad h(0) = 0, \quad h'(0) = \frac{1}{a}$$

<u>proof:</u> Take LT

$$a\left(s^2\tilde{h}(s) - s\cancel{h(0)} - \underbrace{h'(0)}_{1/a}\right) + b\left(s\tilde{h}(s) - \cancel{h(0)}\right) + c\tilde{h}(s) = 0$$

$$(as^2 + bs + c)\tilde{h}(s) = 1$$

$$\tilde{h}(s) = \frac{1}{as^2+bs+c}$$

$\square$

$h(t)$ is known as the (unit) impulse response function

~~ex~~ Using convolution, solve

$$y'' + y = \sin(t), \qquad y(0) = 1, \quad y'(0) = 0$$

$$y = y_h + y_p$$

y_h: $y_h'' + y_h = 0, \quad y_h(0) = 1, \quad y_h'(0) = 0$

$$y_h(t) = \cos(t)$$

y_p: $y_p'' + y_p = \sin(t) \qquad y(0) = 0 = y'(0)$

$$\text{LT} \Rightarrow s^2\tilde{y}_p(s) + \tilde{y}_p(s) = \frac{1}{s^2+1}$$

$$LT \Rightarrow \quad s^2\tilde{y}_p(s) + \tilde{y}_p(s) = \frac{1}{s^2+1}$$

$$\tilde{y}_p(s) = \frac{1}{s^2+1} \cdot \frac{1}{s^2+1}$$

$$\Rightarrow \quad y_p(t) = \sin t * \sin t$$

$$= \int_0^t \sin(t-\tau)\sin(\tau)\,d\tau$$

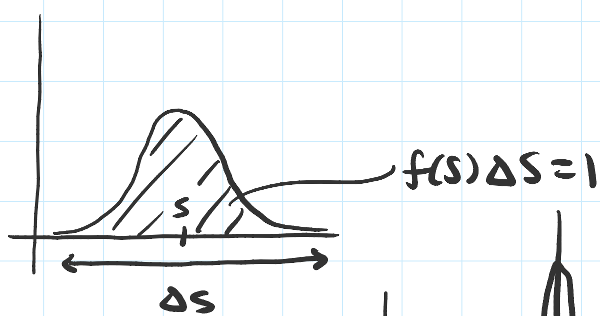
$$= \frac{1}{2}\sin t - \frac{1}{2}t\cos t$$

$$y(t) = y_h(t) + y_p(t)$$

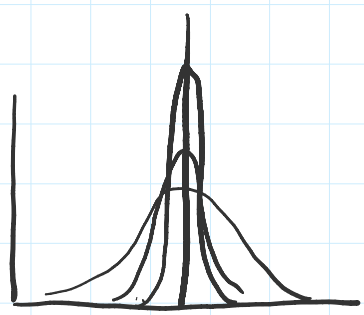
$$= \cos t + \frac{1}{2}\sin t - \frac{1}{2}t\cos t.$$

The Dirac Delta Distribution (7.9)

Consider a function f s.t. $f(s)\Delta s = 1$



$\Delta s \to 0$



Def 1

The Dirac delta distribution $\delta(t)$ is characterized by

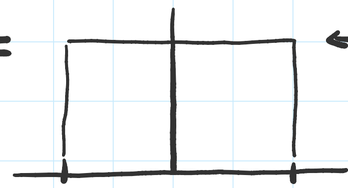
$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

$\delta(t) = \{\infty, \quad t = 0$

and $\displaystyle\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$ for any f cont. in an open interval containing $t = 0$.

<u>Def 2:</u>

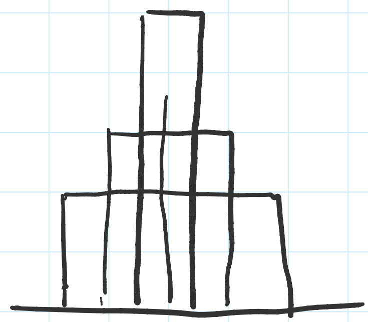
Consider the function

$\delta_{\Delta s}(t) =$ 

$\leftarrow$ height $\dfrac{1}{2\Delta s}$

$\Delta t > 0$

as Δs gets smaller,



$$\delta(t) = \lim_{\Delta s \to 0^+} \delta_{\Delta s}(t)$$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt$$

$$= \lim_{\Delta s \to 0^+} \int_{-\infty}^{\infty} f(t)\,\delta_{\Delta s}(t)\,dt$$

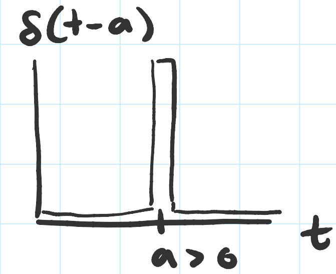
$$= \lim_{\Delta s \to 0^+} \int_{-\Delta s}^{\Delta s} f(t) \cdot \frac{1}{2\Delta s}\,dt$$

$$= \lim_{\Delta s \to 0^+} \Bigg(\underbrace{\frac{1}{2\Delta s} \int_{-\Delta s}^{\Delta s} f(t)\,dt}_{\text{average value } f \text{ on interval}} \Bigg)$$

average value f on interval $[-\Delta s, \Delta s]$

$= f(0)$

mean value theorem



Consider translating by $a > 0$

$$\int_I f(t)\,\delta(t-a)\,dt = f(a)$$

where $a \in I$

What is the LT of δ? $(a>0)$

$$\mathcal{L}(\delta(t-a))(s) = \int_0^\infty e^{-st}\,\delta(t-a)\,dt = e^{-as}$$

$$\mathcal{L}^{-1}(\underbrace{e^{-as}}_{\text{ILT } \delta(t-a)}\ \underbrace{\tilde{f}(s)}_{\text{ILT } f(t)})(t)$$

$= \delta(t-a) * f(t)$

$= \int_0^t \delta(\underbrace{t-a-\tau}_{=0?})\,f(\tau)\,d\underline{\underline{\tau}}$

argument is zero when $\tau = t-a$

I need $0 < t-a \Rightarrow t > a$

$$= \begin{cases} 0, & t < a \\ f(t-a), & t > a \end{cases}$$

$= f(t-a)\,u(t-a)$

Recalling the unit impulse response function $h(t)$ from earlier, I claim if h satisfies

$$\rightarrow a h''(t) + b h'(t) + c h(t) = \delta(t) \qquad h(0) = 0 = h'(0) \leftarrow$$

then it agrees w/ the other def'n of h, i.e.,

$$a(\underline{h * g})'' + b(\underline{h * g})' + c(\underline{h * g}) = g(t)$$

$$(h * g)(t) = \int_0^t h(t - \tau)\, g(\tau)\, d\tau$$

Leibniz' integral rule

$$(h * g)'(t) = \cancel{h(0)} g(t) + \int_0^t h'(t - \tau)\, g(\tau)\, d\tau$$

$$(h * g)''(t) = \cancel{h'(0)} g(t) + \cancel{h(0)} g'(t) + \int_0^t h''(t - \tau)\, g(\tau)\, d\tau$$

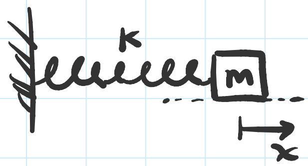
$$a(h * g)''(t) + b(h * g)'(t) + c(h * g)(t)$$
$$= \int_0^t \big(a h''(t - \tau) + b h'(t - \tau) + c h(t - \tau)\big)\, g(\tau)\, d\tau$$
$$= \int_0^t \delta(\underbrace{t - \tau}_{\tau = t})\, g(\tau)\, d\tau = g(t)$$

* The impulse response function h satisfies a (generalized or) distributional differential equation

differential equation

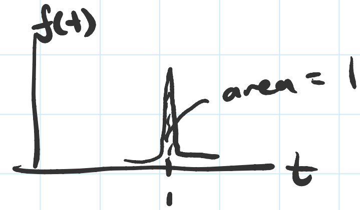
$$ah'' + bh' + ch = \delta(t)$$

ex/ Imagine a mass-spring system



$m = 1$
$k = 1$

Initially, (at time $t=0$) the mass starts at rest in its equilibrium position.
At time $t=1$, a hammer quickly strikes the mass with total impulse 1.
What is the trajectory $x(t)$?



Modelled (to leading order) by the distributional DE

$$x'' = -x + \delta(t-1), \quad x(0) = 0, \quad x'(0) = 0$$

Expect solution should be zero until $t=1$.
Take the LT

$$x'' + x = \delta(t-1)$$

$$s^2 \tilde{x}(s) - s\cancel{x(0)} - \cancel{x'(0)} + \tilde{x}(s) = e^{-s}$$

$$\tilde{x}(s) = \frac{e^{-s}}{s^2+1} \Rightarrow x(t) = u(t-1)\sin(t-1)$$