Power Series Method (8.2, 8.3, 8.4)

f infinitely differentiable,
Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

ex/ Taylor series for e^x at $x_0 = 0$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for example,

$$e^1 = \underbrace{\overset{n=0}{1} + \overset{n=1}{1} + \overset{n=2}{\frac{1}{2}} + \overset{n=3}{\frac{1}{6}}}_{\approx 2.66} + \dots$$

$e \approx 2.7$

$\varepsilon = 0.5$

$\varepsilon = 0.1$

$\varepsilon = 0.00001$

Def: A power series about a point x_0 is an expression $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

We say a power series converges at $x = c$ if $\sum_{n=0}^{\infty} a_n (c-x_0)^n$ converges

i.e. $\lim_{N\to\infty} \sum_{n=0}^{N} a_n (c-x_0)^n = L$ exists

We can define a function $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ whose domain is wherever this series converges.

remark: for every error $\varepsilon > 0$, there exists M
such that

(remark: for every error $\varepsilon > 0$, there exists M such that

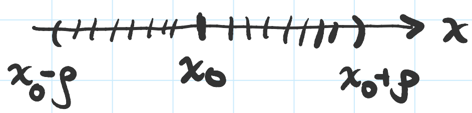
$$\left| L - \sum_{n=0}^{N} a_n (x-x_0)^n \right| < \varepsilon$$

for every $N \geq M$)

Facts:

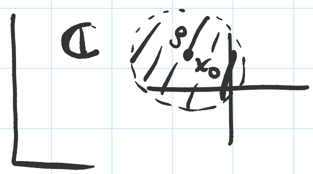
Theorem: (Radius of Convergence)

For a power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$, there exists a number ρ $(0 \leq \rho \leq \infty)$ such that the series converges for $|x-x_0| < \rho$



at $x = x_0$,

$$\sum_{n=0}^{\infty} a_n (x_0 - x_0)^n = 0$$



Ratio Test

If $\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = L$, then $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges with radius of convergence $\rho = L$.

[pf: ratio test series $\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = P$, $0 \leq P < 1$,

then $\sum_{n=0}^{\infty} b_n$ converges.

b_n ← power series

$b_n = a_n (x-x_0)^n$ $\qquad$ $b_{n+1} = a_{n+1} (x-x_0)^{n+1}$

$$\left| \frac{b_{n+1}}{b_n} \right| = \left| \frac{a_{n+1}(x-x_0)}{a_n} \right| = \underbrace{\left| \frac{a_{n+1}}{a_n} \right|}_{\sim 1} |x-x_0| \sim < L \underset{\lim}{<} 1$$

L $\qquad \underbrace{a_n}_{\sim \frac{1}{L}}$ $\qquad$ lim

Vanishing

If $\sum_{n=0}^{\infty} a_n (x-x_0)^n = 0$ in some open interval containing x_0, then $a_n = 0$ for all n.

proof: $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$

$$= a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots$$

$$0 = f(x_0) = a_0$$

$$0 = f'(x_0) = a_1$$

$$\vdots$$

$$0 = f^{(n)}(x_0) = n!\, a_n$$

$\square$

Theorem (Differentiate & Integrate Power Series)

Suppose $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ has radius of convergence $\rho > 0$. Then, f is differentiable and

$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$ has radius of convergence ρ as well.

(Corollary: f is infinitely differentiable and $f^{(k)}$ has radius of convergence ρ

$f^{(k)}(x) = \sum^{\infty} \ldots \frac{d^k}{\ldots} (\ldots x \ldots)^n$ ...)

has radius of convergence ρ

$$f^{(k)}(x) = \sum_{n=k}^{\infty} a_n \frac{d^k}{dx^k}\left[(x-x_0)^n\right]$$

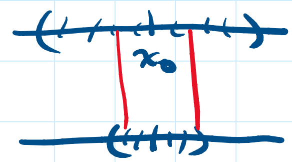
also, f has an antiderivative given by

$$\int f(x)\,dx = \underbrace{\sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1}}_{\text{convergence radius } \rho} + C$$

If two power series about x_0, their sum:

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n + \sum_{n=0}^{\infty} b_n (x-x_0)^n$$

$$= \sum_{n=0}^{\infty} (a_n + b_n)(x-x_0)^n$$



ex/ $(1+x^2)\underline{y''} + \underline{y'} - y = 0$

$y(0) = 1$

$y'(0) = 2$

Expand y in a power series about the initial time; in this case, $x=0$.

$$y(x) = \sum_{n=0}^{\infty} a_n (x - \overset{0}{\overset{\shortparallel}{x_0}})^n = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$1 = y(0) = a_0$

$2 = y'(0) = a_1$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\begin{aligned}
0 &\overset{\downarrow}{=} (1+x^2)\, y''(x) + y'(x) - y(x) \\
&= (1+x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \\
&\qquad + \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1) a_n x^{\underline{n-2}} + \sum_{n=2}^{\infty} n(n-1) a_n x^{\underline{n}} \\
&\qquad + \sum_{n=1}^{\infty} n a_n x^{\underline{n-1}} - \sum_{n=0}^{\infty} a_n x^{\underline{n}}
\end{aligned}$$

$n = k$

Shift summation index

$k = n-2 \qquad n=2 \Leftrightarrow k=0 \qquad \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k$

$k = n-1 \qquad n=1 \Leftrightarrow k=0 \qquad \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=2}^{\infty} k(k-1) a_k x^k \\
&\qquad + \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k - \sum_{k=0}^{\infty} a_k x^k \\
&= \underbrace{\overbrace{2a_2 + a_1 - a_0}^{k=0}}_{=0} + \overbrace{(\underbrace{6a_3 + 2a_2 - a_1}_{=0})}^{k=1} x \\
&\qquad + \sum_{k=2}^{\infty} \Big(\underbrace{(k+2)(k+1)a_{k+2} + k(k-1)a_k + (k+1)a_{k+1} - a_k}_{=0} \Big) x^k
\end{aligned}$$

$k=0: \quad 2a_2 + a_1 - a_0 = 0 \qquad a_2 = \frac{a_0 - a_1}{2} = \frac{1-2}{2} = -1/2$

$k=1 \quad 6a_3 + 2a_2 - a_1 = 0 \qquad a_3 = a_1 - 2a_2 \ldots = 2+1 \ldots = 1/\ldots$

$k=1 \qquad 6a_3 + 2a_2 - a_1 = 0 \qquad a_3 = \frac{a_1 - 2a_2}{6} = \frac{2+1}{6} = 1/2$

$k \geq 2 \qquad (k+2)(k+1)a_{k+2} + k(k-1)a_k$
$+ (k+1)a_{k+1} - a_k = 0$

$$\Rightarrow a_{k+2} = \frac{a_k - (k+1)a_{k+1} - k(k-1)a_k}{(k+1)(k+2)}$$

$\square$

Def: A function is analytic about x_0 if it is given by a power series $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ with a positive radius of convergence.

Consider the DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = \frac{f(x)}{a_2(x)}$$

$(*)$ $y'' + p(x)y' + q(x)y = g(x)$

We say x_0 is a regular (or ordinary) point if p, q, g are analytic at x_0. Otherwise, we say x_0 is a singular point.

Theorem:

Consider the DE $(*)$ and suppose p, q, g are analytic at x_0. Then, there exists are analytic at x_0. Then, there exists a unique solution to IVP (*) with $y(x_0) = y_0$, $y'(x_0) = v_0$, which is analytic

- Furthermore, the radius of convergence is atleast the distance from x_0 to the nearest singular point.

ex $(1+x)y''' + x^2y' + y = e^x$ $\qquad$ $x=0$: $y(0)=0$, $y'(0)=1$, $y''(0)=0$

$$y''' + \frac{x^2}{1+x}y' + \frac{1}{1+x}y = \frac{e^x}{1+x}$$

$x=-1$

x

$\Rightarrow$ Solution is only valid on the interval $(-1,1)$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + \dots$$

$0 = y(0) = a_0$

$1 = y'(0) = a_1$

$0 = y''(0) = 2a_2$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$y'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3}$$

$$(1+x)y''' + x^2y' + y = e^x$$

$$(1+x)\sum_{n=3}^{\infty} n(n-1)(n-2)a_n x^{n-3}$$

$$+ x^2 \sum^{\infty} n a_n x^{n-1} + \sum^{\infty} a_n x^n$$

$$\sum_{n=3}$$

$$+ x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$0 = -\sum_{n=0}^{\infty} \frac{\overbrace{x^n}^{k=n}}{n!} + \sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{\overbrace{n-3}^{k=n-3}}$$

$$+ \sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{\overbrace{n-2}^{k=n-2}}$$

$$+ \sum_{n=1}^{\infty} n a_n \underbrace{x^{n+1}}_{k=n+1} + \sum_{n=0}^{\infty} \underbrace{a_n x^n}_{k=n}$$

$$= -\sum_{k=0}^{\infty} \frac{x^k}{k!} + \sum_{k=0}^{\infty} (k+3)(k+2)(k+1) a_{k+3} x^k$$

$$+ \sum_{k=1}^{\infty} (k+2)(k+1) k a_{k+2} x^k$$

$$+ \sum_{k=2}^{\infty} (k-1) a_{k-1} x^k + \sum_{k=0}^{\infty} a_k x^k$$

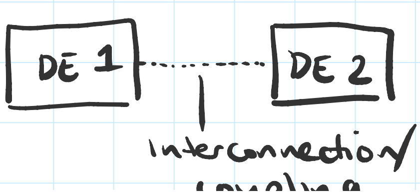
...

Systems of DEs

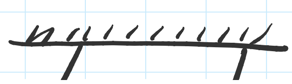
Why consider it?

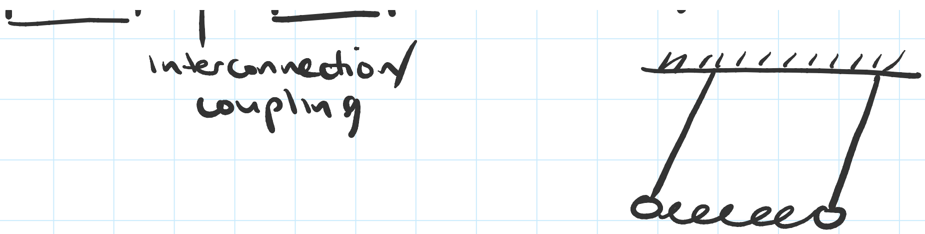
- As we saw in HW1, any n^{th}-order DE can be converted to a 1^{st}-order system of DEs of dimension n.

- Describe interconnection or coupling of DEs



$\Rightarrow$ system of DEs





- Newton's Second Law

$$m\frac{d^2}{dt^2}\vec{x}(t) = \vec{F}\left(t, \vec{x}(t), \frac{d\vec{x}}{dt}\right) \quad \vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

<u>Def</u>:

- A vector-valued function (of time) is a map $\underset{\cap \atop \mathbb{R}}{\vec{x} : I} \longrightarrow \mathbb{R}^n$

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

- A matrix-valued function (of time) is a map $A : I \longrightarrow \mathbb{R}^{n\times n}$

space of $n\times n$ real matrices

$$A(t) = \begin{pmatrix} A_{11}(t) & \cdots & A_{1n}(t) \\ \vdots & \ddots & \vdots \\ A_{n1}(t) & \cdots & A_{nn}(t) \end{pmatrix}$$

Define continuity and differentiability of such functions entrywise

e.g. $$\frac{d}{dt}\vec{x}(t) = \begin{pmatrix} \frac{d}{dt}x_1(t) \\ \vdots \\ \frac{d}{dt}x_n(t) \end{pmatrix}$$

In general, a first-order system of DEs can be written

$$\frac{d}{dt}\vec{x}(t) = \vec{f}(t, \vec{x}(t)) \qquad \vec{f} : \underset{\substack{\uplus \\ t}}{I} \times \underset{\substack{\uplus \\ \vec{x}}}{\mathbb{R}^n} \to \mathbb{R}^n$$

$\vec{f}$ may be a nonlinear function of $\vec{x}$, in which case it may be hard to solve without special assumptions

$\to$ numerically solve on the computer

Linear systems of DEs

Consider the DE

$$(*) \quad \vec{x}'(t) = \underbrace{A(t)}_{\substack{\text{matrix-valued} \\ \text{function } I \to \mathbb{R}^{n \times n}}} \vec{x}(t) + \underbrace{\vec{g}(t)}_{\text{inhomogeneity } I \to \mathbb{R}^n}$$

An IVP is $(*)$ with an initial condition $\vec{x}(t_0) = \vec{x}_0$ for some given $t_0 \in I$ and some given $\vec{x}_0 \in \mathbb{R}^n$.

Theorem: Suppose A and $\vec{g}$ are continuous on an interval I. Then, there exists a unique solution to the IVP, $(*)$ with $\vec{x}(\underset{\substack{\uparrow \\ I}}{t_0}) = \vec{x}_0$. $\square$

Consider the homogeneous case $(\vec{g} \equiv 0)$

$(*) \quad \frac{d}{dt}\vec{x}(t) = A(t)\vec{x}(t) \qquad (\ldots \vec{x} : I \to \mathbb{R}^n)$

$$(*) \quad \frac{d}{dt}\vec{x}(t) = A(t)\,\vec{x}(t) \qquad (\text{dimension } n,\ \vec{x}: I \to \mathbb{R}^n)$$

If $\vec{x}_1$ and $\vec{x}_2$ are solutions to $(*)$, then so is $c_1\vec{x}_1 + c_2\vec{x}_2$ for constants $c_1, c_2 \in \mathbb{R}$.

<u>proof</u>:

$$\begin{aligned} &\frac{d}{dt}\left(c_1\vec{x}_1(t) + c_2\vec{x}_2(t)\right) \\ &= c_1\frac{d}{dt}\vec{x}_1(t) + c_2\frac{d}{dt}\vec{x}_2(t) \\ &= c_1A(t)\vec{x}_1(t) + c_2A(t)\vec{x}_2(t) \\ &= A(t)\left(c_1\vec{x}_1(t) + c_2\vec{x}_2(t)\right) \end{aligned}$$

$\square$

How to solve IVP?

Suppose we have n linearly independent solutions to $(*)$, $\vec{x}_1, \dots, \vec{x}_n$.

<u>Def</u>: We say m vector-valued functions $\vec{x}_1, \dots, \vec{x}_m$ defined on I are linearly dependent if there exists constants $c_1, \dots, c_m$, not all zero, s.t.

$$c_1\vec{x}_1(t) + \dots + c_m\vec{x}_m(t) = 0 \text{ for all } t \in I.$$

Otherwise, say they are linearly independent

<u>Theorem</u>: Let $\vec{x}_1, \dots, \vec{x}_n$ be lin. ind. solutions to $(*)$. Then, any solution to the IVP $(*)$ with $\vec{x}(t_0) = \vec{x}_0$ can be expressed

$$c_1\vec{x}_1(t) + \dots + c_n\vec{x}_n(t)$$

$$c_1\vec{x}_1(t) + \dots + c_n\vec{x}_n(t)$$

for the appropriate choices of $c_1, \dots, c_n$.

<u>proof:</u>

Can we choose $c_1, \dots, c_n$ s.t.

$$c_1\vec{x}_1(t_0) + \dots + c_n\vec{x}_n(t_0) = \vec{x}_0 \ ?$$

$$\iff \underbrace{\begin{pmatrix} | & | & & | \\ \vec{x}_1(t_0) & \vec{x}_2(t_0) & \cdots & \vec{x}_n(t_0) \\ | & | & & | \end{pmatrix}}_{n\times n \text{ matrix}} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \vec{x}_0$$

Is this matrix invertible

$$W(\vec{x}_1(t), \dots, \vec{x}_n(t)) = \det\begin{pmatrix}\vec{x}_1(t) & \cdots & \vec{x}_n(t)\end{pmatrix}$$

We know $\uparrow$ is nonzero for some $t \in I$.

Is it nonzero for all $t \in I$? Suppose, for contradiction, t_*, s.t. $W(\vec{x}_1(t_*), \dots, \vec{x}_n(t_*)) = 0$

$\iff \exists$ constants $d_1, \dots, d_n$ not all zero s.t.

$$d_1\vec{x}_1(t_*) + \dots + d_n\vec{x}_n(t_*) = 0 \leftarrow \vec{0} \in \mathbb{R}^n$$

The function

$$\vec{x}(t) = d_1\vec{x}_1(t) + \dots + d_n\vec{x}_n(t)$$

is a solution to the IVP $\vec{x}(t_*) = 0$

but so is $\vec{y}(t) = \vec{0}$

$$\vec{x}(t) = 0 \text{ everywhere on } I \quad \rightarrow\leftarrow$$

$\Rightarrow$ the Wronskian is nonzero for all $t \in I$

$$\Rightarrow \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \vec{x}_1(t_0) & \cdots & \vec{x}_n(t_0) \end{pmatrix}^{-1} \vec{x}_0$$

$\square$

Constant Matrix

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t), \quad A \in \mathbb{R}^{n \times n}, \quad \vec{x} : I \to \mathbb{R}^n$$

simple example $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\lambda \in \mathbb{R}$

$$\vec{x} = \begin{pmatrix} u \\ v \end{pmatrix} \qquad \begin{pmatrix} du/dt \\ dv/dt \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda_1 u & 0 \\ 0 & \lambda_2 v \end{pmatrix}$$

$$du/dt = \lambda_1 u$$
$$dv/dt = \lambda_2 v$$

off-diagonal terms in a matrix describe the coupling between each equation

$$\Rightarrow u(t) = c_1 e^{\lambda_1 t}$$
$$v(t) = c_2 e^{\lambda_2 t}$$

$$\vec{x}(t) = c_1 \underbrace{e^{\lambda_1 t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{x}_1(t)} + c_2 \underbrace{e^{\lambda_2 t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{x}_2(t)}$$

$$W(x_1(t), x_2(t)) = \det\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} = e^{(\lambda_1 + \lambda_2)t} \neq 0 \; \checkmark$$

Observe $A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\lambda_1\\0\end{pmatrix} = \lambda_1\begin{pmatrix}1\\0\end{pmatrix}$