Math 20D Lecture 9: First-order Linear Systems cont.

Tuesday, July 26, 2022 2:53 PM

- Homework 3 solutions posted
- MATLAB Quiz tomorrow (see Canvas announcement for details)
- Homework 4 due Thursday 11:59 pm
- If you have time, I'd appreciate it if you fill out your CAPEs (course and professor evaluations). As a newer instructor, I really appreciate the feedback. It is completely anonymous and I can't even access the evaluations until after grades are submitted anyway. The only thing I can access beforehand is how many students completed CAPEs; if more than 60% of the class completes it, I will add on 0.5% extra credit.
- Optional lecture next Thursday. I think it'll be the most fun lecture and I'll have some demonstrations as well (for both numerical differential equations and partial differential equations), so I hope you can make it! I think it will be a good lecture to synthesize what we discussed in the course and have a glimpse at what's beyond.
- Final exam is in-person; on Saturday July 30th from 7 pm to 10 pm in SOLIS 104

<u>Def</u>: Given an $n\times n$ matrix $A \in \mathbb{R}^{n\times n}$, we say $0 \neq \vec{u} \in \mathbb{C}^n$ is an eigenvector of A with associated eigenvalue $\lambda \in \mathbb{C}$ if

$$A\vec{u} = \lambda \vec{u}$$

$$0 \neq c\vec{u}: \quad A c\vec{u} = cA\vec{u} = c\lambda\vec{u} = \lambda(c\vec{u})$$

$$\iff A\vec{u} - \lambda\vec{u} = 0$$

$$\iff (A - \lambda \mathbb{1})\vec{u} = 0$$

$\mathbb{1}$: $n\times n$ identity matrix

$B : \mathbb{R}^k \to \mathbb{R}^m$ linear, $B \in \mathbb{R}^{m\times k}$

$\ker B = \{\vec{b} \in \mathbb{R}^k : B\vec{b} = \vec{0}\}$

$\iff$ the $n\times n$ matrix $A - \lambda\mathbb{1}$ has a nontrivial kernel

$$0 \neq \vec{u} \in \ker(A - \lambda\mathbb{1})$$

$$\iff \det(A - \lambda\mathbb{1}) = 0 \quad \text{characteristic equation for } A$$

Consider the DE

$$(*) \quad \frac{d}{dt}\vec{x} = A\vec{x}$$

Suppose we know $(\vec{u}, \lambda)$ is an eigenpair of A (eigenvector, eigenvalue)

(*) $\frac{d}{dt} x = A x$

Suppose we know $(\vec{u}, \lambda)$ is an eigenpair of A.
(the first slot is the eigenvector, the second the eigenvalue)

I claim $e^{\lambda t}\vec{u}$ solves (*)

$$\frac{d}{dt}(e^{\lambda t}\vec{u}) = \lambda e^{\lambda t}\vec{u}$$

$$A(e^{\lambda t}\vec{u}) = e^{\lambda t} A\vec{u} = \lambda e^{\lambda t}\vec{u} \quad \checkmark$$

This is only one solution. For $A \in \mathbb{R}^{n\times n}$, we expect n linearly independent solutions.

Facts from linear algebra.

<u>Thm</u>: Suppose $(\vec{u}_1, \lambda_1), \ldots, (\vec{u}_K, \lambda_K)$ are K eigenpairs of $A \in \mathbb{R}^{n\times n}$ $(K \le n)$ and all of the eigenvalues are distinct.
Then, $\{\vec{u}_1, \ldots, \vec{u}_K\}$ are linearly independent

<u>proof</u>:

Suppose they aren't linearly ind.

One of the eigenvectors is a nontrivial combination of the others, say $\vec{u}_K$

$$\vec{u}_K = \sum_{j=1}^{K-1} c_j \vec{u}_j \qquad \text{not all of } c_j \text{ equal } 0$$

$$A\vec{u}_K = \lambda_K \vec{u}_K = \sum_{j=1}^{K-1} c_j \lambda_K \vec{u}_j$$

$$\sum_{j=1}^{K-1} c_j A\vec{u}_j = \sum_{j=1}^{K-1} c_j \lambda_j \vec{u}_j$$

$$\sum_{j=1} c_j A \vec{u}_j = \sum_{j=1} c_j \lambda_j \vec{u}_j$$

$$0 = \sum_{j=1}^{k-1} \underbrace{c_j (\lambda_j - \lambda_k)}_{=0 \text{ for all } j} \underline{\underline{\vec{u}_j}}$$

$\Rightarrow$ there exists j, $1 \le j \le k-1$, s.t.
$\lambda_j = \lambda_k$.

$\square$

careful:
The converse is not true $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

eigenval
$\lambda_1 = 1$ with $\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 1$ with $\vec{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

ex/ Every symmetric matrix $A = A^T$, $\left(\text{ex } \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\right)$
$A \in \mathbb{R}^{n \times n}$, has n real eigenvalues
and n linearly independent eigenvectors.

ex/ Find the general solution for

$$\frac{d}{dt}\vec{x} = A\vec{x}, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A - \lambda \mathbb{1} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}$$

char. eqn is

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$$0 = \det(A - \lambda \mathbb{1}) = (1-\lambda)^2 - 1$$ quadratic polynomial in λ

$$\Rightarrow \lambda = 1 \pm 1 = 0 \text{ or } 2$$

$$\lambda_1 = 0, \quad \lambda_2 = 2$$

eigenvec $\vec{u}_1$ eigenvec $\vec{u}_2$

$$(A - \lambda_1 \mathbb{1})\vec{u}_1 = 0$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} u_{11} + u_{12} \\ u_{11} + u_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Choose one value $u_{12} = 1$

$\Rightarrow u_{11} = -u_{12} = -1$

$\Rightarrow \vec{u}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$(A - \lambda_2 \mathbb{1})\vec{u}_2 = 0$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_{21} \\ u_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Choose $u_{22} = 1$

$u_{21} = 1$

$\Rightarrow \vec{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

General solution is

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{u}_1 + c_2 e^{\lambda_2 t}\vec{u}_2 \leftarrow$$

Why lin. ind? $\vec{x}(0) = \vec{x}_0 \in \mathbb{R}^2$

$$\vec{x}_0 = \vec{x}(0) = c_1\vec{u}_1 + c_2\vec{u}_2$$

$$= \underbrace{\begin{pmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{pmatrix}}_{P}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$\underbrace{\begin{pmatrix} | & | \end{pmatrix}}_{P}$ invertible $\iff$ $\vec{u}_1, \vec{u}_2$ are lin. ind.

ex $ax'' + bx' + cx = 0 \quad (a \neq 0)$

$x'' + \frac{b}{a}x' + \frac{c}{a}x = 0$

$x' = v$

$v' = x'' = -\frac{b}{a}v - \frac{c}{a}x$

$$\frac{d}{dt}\underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_{\vec{y}} = \begin{pmatrix} v \\ -\frac{b}{a}v - \frac{c}{a}x \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_{y}$$

$$\frac{d}{dt}\vec{y} = A\vec{y}$$

Eigenvalues of A

$$0 = \det(A - \lambda \mathbb{1}) = \det\begin{pmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{pmatrix}$$

$$= \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a}$$ roots from quadratic formula

$$\lambda_\pm = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$ assume $b^2 - 4ac \neq 0$

eigenvectors $\vec{u}_\pm$

$$(A - \lambda_\pm \mathbb{1})\vec{u}_\pm = 0$$

$$\begin{pmatrix} -\lambda_\pm & 1 \\ -\frac{c}{\cdot} & -\frac{b}{\cdot} - \lambda_\pm \end{pmatrix}\begin{pmatrix} u_{\pm 1} \\ u_{\pm 2} \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} -\lambda_\pm & 1 \\ -\frac{c}{a} & -\frac{b}{a}-\lambda_\pm \end{pmatrix}\begin{pmatrix} u_{\pm 1} \\ u_{\pm 2}\end{pmatrix} = \vec{0}$$

$$u_{\pm 1} = 1 \Rightarrow u_{\pm 2} = \lambda_\pm \qquad \vec{u}_\pm = \begin{pmatrix} 1 \\ \lambda_\pm \end{pmatrix}$$

general solution

$$\vec{y}(t) = c_1 e^{\lambda_+ t}\begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} + c_2 e^{\lambda_- t}\begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{\lambda_+ t} + c_2 e^{\lambda_- t} \\ c_1 \lambda_+ e^{\lambda_+ t} + c_2 \lambda_- e^{\lambda_- t} \end{pmatrix} = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$$

$\square$

Complex eigenvalues/vectors (9.6)

Eigenvalues arise as roots to polynomial

$\Rightarrow$ Complex eigenvalues ($\mathrm{Im}(\lambda) \neq 0$) come in conjugate pairs

$\mathbb{R}^{n\times n}$

Let $\lambda = \alpha + i\beta$ be an eigenvalue of A with eigenvector $\vec{u} = \vec{a} + i\vec{b}$

Then, $\lambda^* = \alpha - i\beta$ is an eigenvalue of A with eigenvector $\vec{u}^* = \vec{a} - i\vec{b}$

<u>Proof:</u>

$$A\vec{u} = \lambda\vec{u}$$

$$(A\vec{u})^* = (\lambda\vec{u})^* \qquad A = A^*$$

$$A\vec{u}^* = \lambda^*\vec{u}^*$$

$\square$

$$A\vec{u}^* = \lambda^* \vec{u}^* \qquad \square$$

For a 2×2 matrix w/ complex eigenvalues, have general solution

$$\lambda = \alpha + i\beta, \quad \vec{u} = \vec{a} + i\vec{b}$$

$$\vec{x}(t) = c_1 e^{\lambda t}\vec{u} + c_2 e^{\lambda^* t}\vec{u}^*$$

$$= c_1 e^{\alpha t} e^{i\beta t}(\vec{a} + i\vec{b}) + c_2 e^{\alpha t} e^{-i\beta t}(\vec{a} - i\vec{b})$$

$$e^{i\beta t}(\vec{a} + i\vec{b}) = (\cos\beta t + i\sin\beta t)(\vec{a} + i\vec{b})$$

$$= (\cos(\beta t)\vec{a} - \sin(\beta t)\vec{b}) + i(\sin(\beta t)\vec{a} + \cos(\beta t)\vec{b})$$

$$e^{-i\beta t}(\vec{a} - i\vec{b}) = (\cos(\beta t) - i\sin(\beta t))(\vec{a} - i\vec{b})$$

$$= (\cos(\beta t)\vec{a} - \sin(\beta t)\vec{b}) - i(\sin(\beta t)\vec{a} + \cos(\beta t)\vec{b})$$

Real form of gen. solution

$\Rightarrow$

$$\vec{x}(t) = d_1 e^{\alpha t}(\cos(\beta t)\vec{a} - \sin(\beta t)\vec{b}) + d_2 e^{\alpha t}(\sin(\beta t)\vec{a} + \cos(\beta t)\vec{b})$$

<u>Def</u>: A matrix $A \in \mathbb{R}^{n\times n}$ is called diagonalizable if there exists an invertible matrix $P \in \mathbb{C}^{n\times n}$ and a diagonal matrix $D \in \mathbb{C}^{n\times n}$ s.t.

$$A = \underline{PDP^{-1}}$$

$\Leftrightarrow$ A has n linearly independent eigenvectors $\vec{u}_1, \dots, \vec{u}_n$ & eigenvalues $\lambda_1, \dots, \lambda_n$

$\vec{u}_1, \dots, \vec{u}_n$ & eigenvalues $\lambda_1, \dots, \lambda_n$

$$P = \begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{pmatrix} \quad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$Ax = \underbrace{P \underbrace{D \underbrace{P^{-1} x}}}$$

$$\frac{d}{dt} x = Ax$$

$$\frac{d}{dt} x = PDP^{-1}x$$

$$\frac{d}{dt}(\underbrace{P^{-1}x}_{\vec{y}}) = D(\underbrace{P^{-1}x}_{\vec{y}})$$

$$\frac{d}{dt} \vec{y} = D\vec{y}$$

Eigenvectors/values of A in $\frac{d}{dt}\vec{x} = A\vec{x}$ describe fundamental modes of the system. They allow us to decompose complicated dynamics into linear superpositions of decoupled (or noninteracting) modes.

decoupled (or noninteracting) modes.

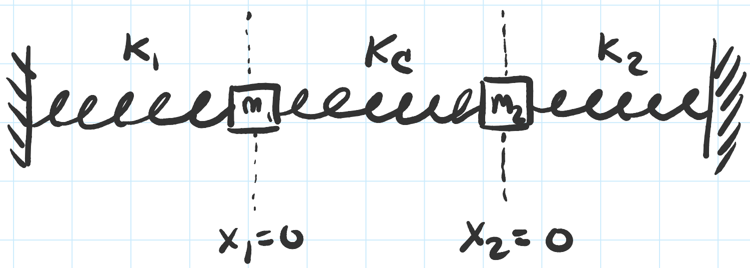
[Variation of parameters 9.7]

[What if a matrix is not diagonalizable?
section 9.8 matrix exponential]

$$\frac{d}{dt} x = \lambda x \iff x = e^{\lambda t}$$

$$\frac{d}{dt} \vec{x} = A \vec{x} \iff \vec{x} = e^{At}$$

ex/



ma = F

$$m_1 x_1'' = -k_1 x_1 + k_c(x_2 - x_1)$$

$$m_2 x_2'' = -k_2 x_2 - k_c(x_2 - x_1)$$

First-order system of dim 4

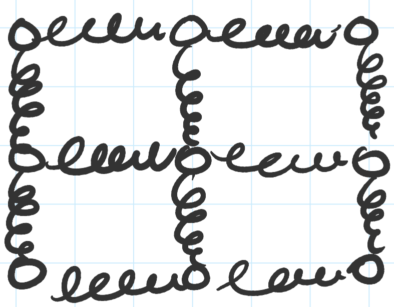
$$v_1 = x_1'$$

$$v_2 = x_2'$$

$$\vec{y}(t) = \begin{pmatrix} x_1(t) \\ v_1(t) \\ x_2(t) \\ v_2(t) \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k_1+k_c)}{m_1} & 0 & k_c/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k_c/m_2 & 0 & -\frac{(k_2+k_c)}{m_2} & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{pmatrix}$$

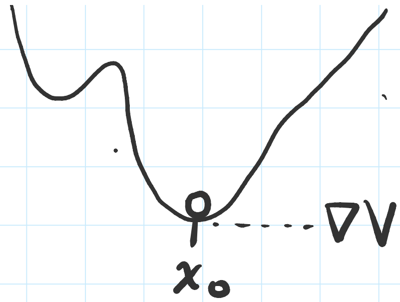
- We use such systems to describe lattices of atoms or molecules where the "springs" are the atomic forces or molecular bonds



- Why does this approximation work?

$$m\vec{x}'' = -\nabla V(\vec{x}) \qquad \nabla V = \left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right)^T$$

$V(x)$



$\nabla V(\vec{x}_0) = 0$

$$V(\vec{x}_0 + \Delta\vec{x})$$
$$= V(\vec{x}_0) + \cancel{\nabla V(\vec{x}_0)}^{0} \cdot (\vec{x} - \vec{x}_0)$$
$$+ \sum_{i} \frac{1}{2} \partial_i \partial_j V(\vec{x}_0) \cdot (x_i - x_{0i})(x_j - x_{0j})$$
$$+ \mathcal{O}(\Delta x^3)$$

In the 1d case, this looks like $V'(x_0) = 0$ and

$$\Rightarrow V(x_0 + \Delta x) = V(x_0) + \cancel{V'(x_0)}^{0} \Delta x + \frac{1}{2} V''(x_0) \Delta x^2 + \mathcal{O}(\Delta x^3)$$

$$\Rightarrow V'(x) = \underbrace{V''(x_0)}_{>0 \text{ since } x_0 \text{ is a local min}} (x - x_0) + \mathcal{O}(\Delta x)^2$$

$$\Rightarrow m x'' = -V'(x) = -\underbrace{V''(x_0)}_{>0 \text{ so looks like a spring constant}} (x - x_0) + \mathcal{O}(\Delta x)^2$$

$\Rightarrow$ Newton's law about a local min. in the potential looks like a harmonic oscillator, to first-order.