

Lecture 11 - Parametrized Surfaces

- Asynchronous lecture replacing the lecture on Monday 10/18
- Read section 7.3 in the text

We know how to integrate over:

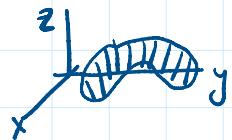
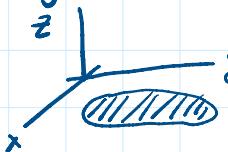
Intervals (single var. calculus)

Areas $\subseteq \mathbb{R}^2$ (double integrals)

Volume $\subseteq \mathbb{R}^3$ (triple integrals)

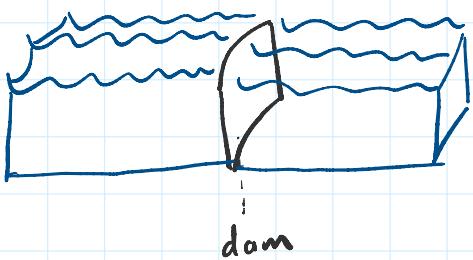
Paths $\subseteq \mathbb{R}^n$ (path & line integrals)

Why integrate over surface?



- Surfaces arise naturally in physical situations.

For example, imagine building a dam



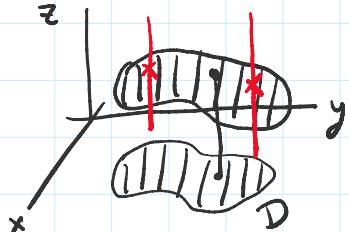
Water exerts a pressure = $\frac{\text{force}}{\text{unit area}}$,
on the dam.

Total force = adding up pressure
(times small amount of area) over
the whole surface
→ surface integral.

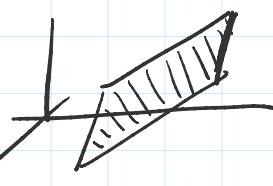
Surfaces

ex/ Graph of a function $z = f(x, y)$, $f: D \rightarrow \mathbb{R}$

$$\{(x, y, f(x, y)) : (x, y) \in D\}$$

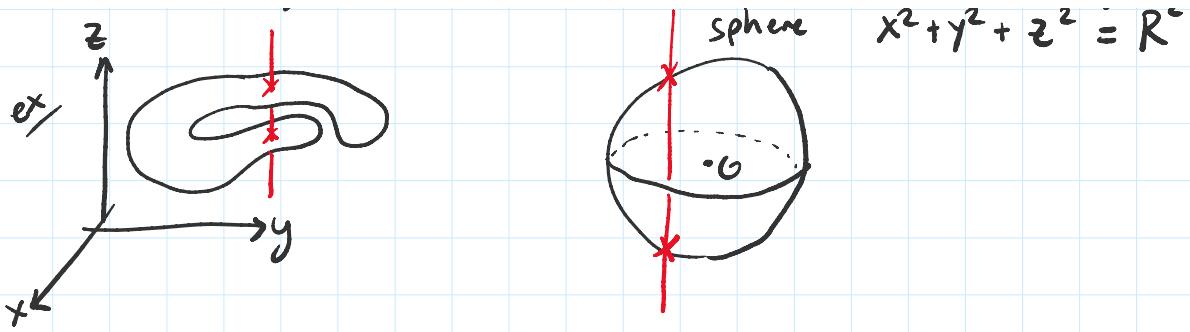


ex/ Plane



sphere $x^2 + y^2 + z^2 = R^2$

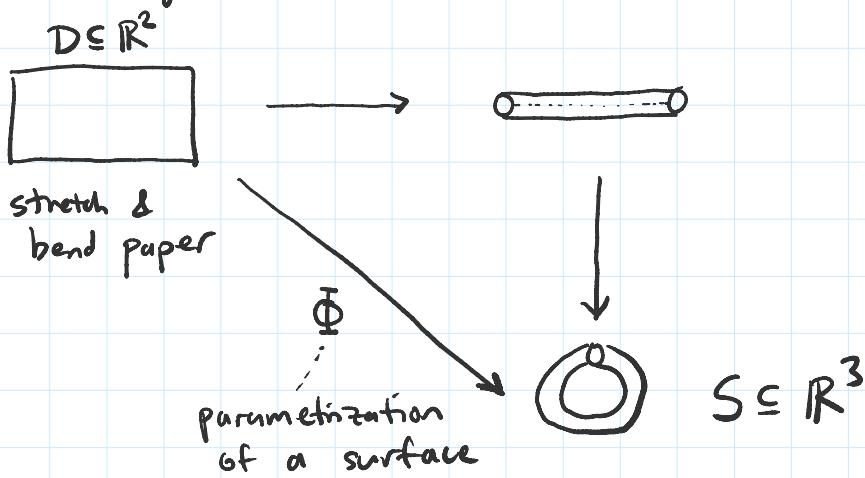
2-dim. surface
 $\subseteq \mathbb{R}^3$



ex Torus (or donut)



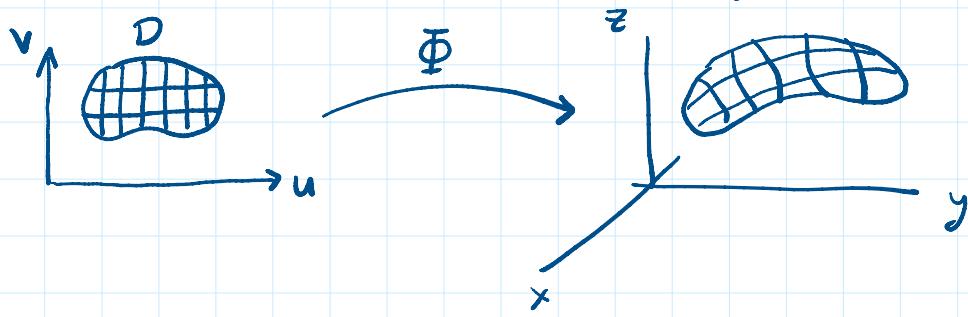
How to get a torus?



Def: A parametrization of a surface is a map $\Phi : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$, where D is a domain in \mathbb{R}^2 .
The surface S is defined as the image of Φ , $S = \Phi(D)$.

Use coordinates $(u, v) \in D$, $(x, y, z) \in \mathbb{R}^3$

$$\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$$



- We'll usually assume that Φ is continuously differentiable (C^1).

ex/ Parametrize the surface of a sphere

$$\mathbb{S}_R^2 = \{(x, y, z) : x^2 + y^2 + z^2 = R^2\}$$

How do we parametrize the ball? $x^2 + y^2 + z^2 \leq R^2$

Spherical coordinates

$$(x, y, z) \Big|_{(r, \theta, \phi)} = (r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi)$$

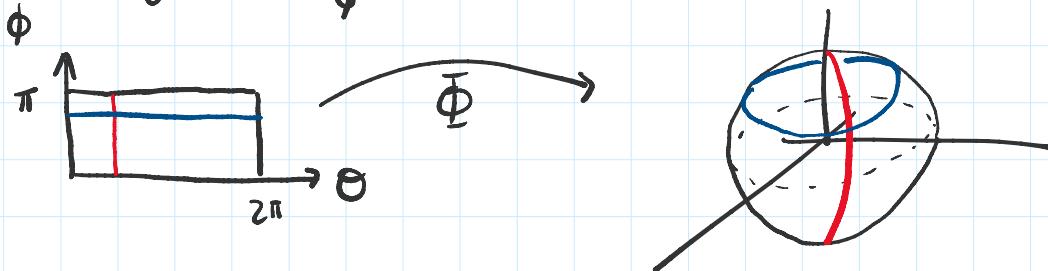
$$0 \leq r \leq R \quad 0 \leq \phi \leq \pi.$$

$$0 \leq \theta < 2\pi$$

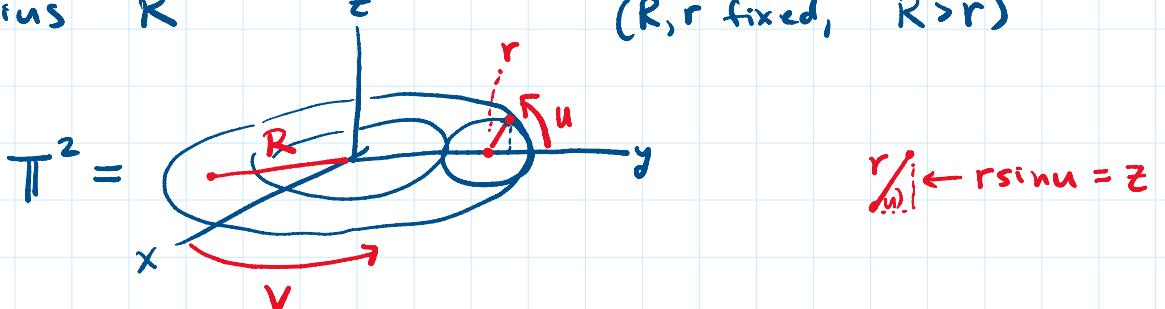
For \mathbb{S}_R^2 , restrict $r = R$. $\theta \in [0, 2\pi]$, $\phi \in [0, \pi]$

$$\Phi(\theta, \phi) = (\underbrace{R \cos \theta \sin \phi}_{x(\theta, \phi)}, \underbrace{R \sin \theta \sin \phi}_{y(\theta, \phi)}, \underbrace{R \cos \phi}_{z(\theta, \phi)})$$

$$\Phi: \underbrace{[0, 2\pi]}_{\theta} \times \underbrace{[0, \pi]}_{\phi} \rightarrow \mathbb{S}^2 \subset \mathbb{R}^3$$



ex/ Torus w/ inner radius r and outer radius R (R, r fixed, $R > r$)

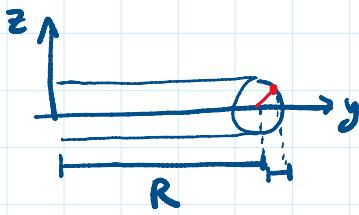


$$\Phi(u, v) = ((R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u)$$

$$= ((R + r\cos u) \underline{\cos v}, (R + r\cos u) \underline{\sin v}, r \sin u)$$

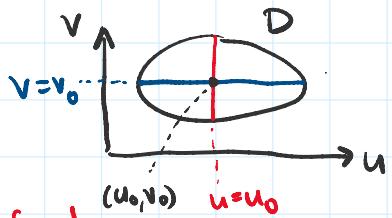
$u \in [0, 2\pi], v \in [0, 2\pi]$

$R + r\cos u$?



[Tangent & Normal Vectors to Surfaces]

Param. of surface



$$\vec{\Phi}(u_0, v) \quad v \in [a, b]$$

$$\vec{T}_v(u_0, v_0) = \left(\frac{\partial}{\partial v} \vec{\Phi}(u_0, v) \right)_{v=v_0}$$

$$= \left(\frac{\partial x}{\partial v} \Big|_{(u_0, v_0)}, \frac{\partial y}{\partial v} \Big|_{(u_0, v_0)}, \frac{\partial z}{\partial v} \Big|_{(u_0, v_0)} \right)$$

varying
fixed

$$\vec{\Phi}(u, v_0)$$

$$\vec{T}_u(u_0, v_0) = \left(\frac{\partial}{\partial u} \vec{\Phi}(u, v_0) \right)_{u=u_0}$$

$$= \left(\frac{\partial x}{\partial u} \Big|_{(u_0, v_0)}, \frac{\partial y}{\partial u} \Big|_{(u_0, v_0)}, \frac{\partial z}{\partial u} \Big|_{(u_0, v_0)} \right).$$

Normal vector at $\vec{\Phi}(u_0, v_0)$ is

$$\vec{n}_+(u_0, v_0) = \vec{T}_u(u_0, v_0) \times \vec{T}_v(u_0, v_0)$$

$$\vec{n}_+ = -\vec{n}_-$$

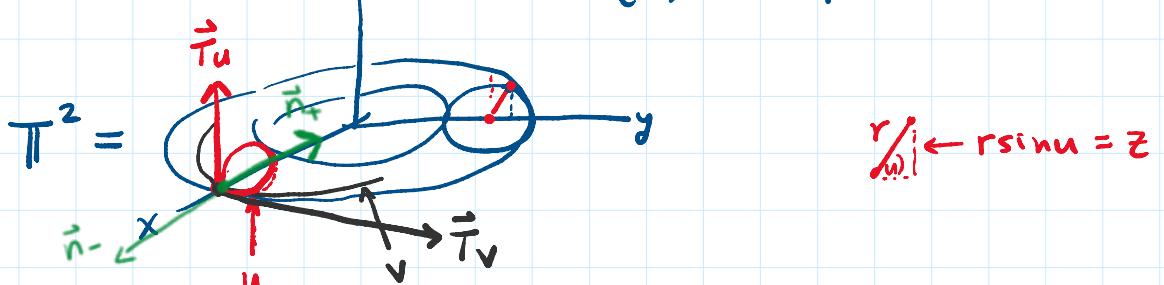
$$\vec{n}_-(u_0, v_0) = \vec{T}_v(u_0, v_0) \times \vec{T}_u(u_0, v_0)$$

We say a surface is regular at $\vec{\Phi}(u_0, v_0)$ if
 $\vec{n}(u_0, v_0) \neq 0$.

We say a surface is regular at (u_0, v_0) if
 $\vec{n}(u_0, v_0) \neq 0$.

If the surface is regular at all points on it,
we call S a regular surface.

ex/ Torus w/ inner radius r and outer
radius R (R, r fixed, $R > r$)



$$\begin{aligned}\Phi(u, v) &= ((R+r\cos u)\cos v, (R+r\cos u)\sin v, r\sin u) \\ &= (x(u, v), y(u, v), z(u, v))\end{aligned}$$

$u \in [0, 2\pi], v \in [0, 2\pi]$

Compute tangent and normal vectors to T^2
at the point $(R+r, 0, 0)$

$$u=0, v=0 \leftarrow (u_0, v_0) = (0, 0)$$

$$\Phi(0, 0) = (R+r, 0, 0)$$

$$\vec{T}_u(0, 0) = \frac{\partial}{\partial u} \Phi \Big|_{(0, 0)}$$

$$= (-r\sin u \cos v, -r\sin u \sin v, r\cos u) \Big|_{(0, 0)} = (0, 0, r)$$

$$\vec{T}_v(0, 0) = \frac{\partial}{\partial v} \Phi \Big|_{(0, 0)}$$

$$= (- (R+r\cos u) \sin v, (R+r\cos u) \cos v, 0) \Big|_{(0, 0)}$$

$$= (0, R+r, 0)$$

points in \hat{y} direction

$$\begin{aligned}\vec{n}_+(0, 0) &= \vec{T}_u(0, 0) \times \vec{T}_v(0, 0) \\ &= |\hat{i} \quad \hat{j} \quad \hat{k}| \quad \begin{matrix} \nearrow & \searrow \\ \downarrow & \uparrow \end{matrix}\end{aligned}$$

$$\mathbf{N} + (\mathbf{v}, \mathbf{v}) = \mathbf{i}_{\mathbf{v}(\mathbf{v}, \mathbf{v})} \wedge \mathbf{j}_{\mathbf{v}(\mathbf{v}, \mathbf{v})}$$
$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & r \\ 0 & R+r & 0 \end{vmatrix} = -r(R+r) \hat{\mathbf{i}}$$
$$= (-r(R+r), 0, 0)$$